NO-FREE-LUNCH EQUIVALENCES FOR EXPONENTIAL LÉVY MODELS UNDER CONVEX CONSTRAINTS ON INVESTMENT

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Abstract. We provide equivalence of numerous no-free-lunch type conditions for financial markets where the asset prices are modeled as exponential Lévy processes, under possible convex constraints in the use of investment strategies. The general message is the following: if any kind of free lunch exists in these models it has to be of the most egregious type, generating an *increasing* wealth. Furthermore, we connect the previous to the existence of the *numéraire portfolio*, both for its particular expositional clarity in exponential Lévy models and as a first step in obtaining analogues of the no-free-lunch equivalences in general semimartingale models, a task that is taken on in Karatzas and Kardaras [21].

0. Introduction

0.1. **Discussion.** An *exponential Lévy process* — as its name suggests — is simply the exponential of a Lévy process. Models of financial markets that assume an exponential Lévy structure for the movement of the stock-price processes have become increasingly popular in the last years, partly because of their analytical tractability (since their distributional properties are uniquely determined by their Lévy triplet) and partly because they provide a reasonably good fit to actual financial data. Noteworthy examples are the four-parameter *CGMY* model of Carr, Geman, Madan and Yor [4] and the *hyperbolic model* of Eberlein, Keller and Prause [13]. One effect of this popularity is the proliferation of academic courses that include in their teaching curriculum models of this sort.

It is somewhat of *folklore* that if free lunches exist in exponential Lévy models, they are of the most egregious form: one can invest in a way so to obtain an *increasing* wealth process. As a result, many subtle differences existing in different formulations of a "free lunch" definition in more general models disappear, something with both good and bad consequences. On the positive side, one can provide a proof of the Fundamental Theorem of Asset Pricing (FTAP) with minimal effort that can be easily taught — in particular, no functional-analytic background is required and the proof uses reasonably standard facts from Lévy-process theory. The offset is that mere knowledge of the no-free-lunch situation in exponential Lévy models is inadequate to provide the whole picture and complications that prevail in semimartingale models.

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The purpose of this paper is twofold: first, to provide a quick and easy proof[1] of the above folklore fact for multi-asset, finite-time horizon models under convex constraints, establishing many equivalences regarding no-free-lunch notions and (super)martingale measures; second, to explore the structure of the so-called *numéraire portfolio* in the context of exponential Lévy models to the extend where the transition to general semimartingale models will be possible.

Economic agents typically face restrictions in the free use of portfolios — a first example being short-sale constraints. In this "constrained" setting and in the context of the FTAP, one cannot claim any more that no-free-lunch criteria expressed in terms of the restricted collection of admissible strategies imply existence of equivalent martingale measures for the stock-price process. An extreme example is total prevention for the use of any portfolio, except keeping all the wealth in the savings account; in this trivial case even if free lunches exist in the *unconstrained* market, they cannot be used because the agent *cannot* invest in them. To compensate for the fact that we are *only* considering constrained strategies, we have to introduce notions of equivalent probability measures that act only on the wealth-process class and not on the stock-price process. As long as the constraints are of the form of a convex *cone*, the concept of *equivalent supermartingale measure* (ESMM — also called *separating measure* in the literature) does the trick: we have to make sure that under an equivalent change of probability all wealth processes are supermartingales.

The first main result of the paper — a version of the FTAP for convex-cone-constrained exponential Lévy models — is Theorem 2.7. The "difficult" part of the proof of Theorem 2.7 follows the idea of Rogers [24]: solve a utility maximization problem and construct an ESMM using the marginal utility evaluated on the optimal wealth process as density. Rogers implemented this for the discrete-time case; an inductive construction based on the simple static one-time-period model had to be utilized in order to fully prove the FTAP for multi time-period models. Unfortunately, this construction does not carry to general continuous-time models, exactly because this inductive step cannot be carried over. Nevertheless, one can use this idea when Lévy processes are involved, because of their "independent and stationary increments" structure. Theorem 2.7 presents seven equivalences involving equivalent supermartingale measures that respect the exponential Lévy structure, several no-free-lunch notions and — most importantly — a condition that involves only the characteristic triplet of the generating Lévy process. Let us note that simpler statements than that of Theorem 2.7, dealing only with equivalences for the one-stock, unconstrained case, have already appeared in Jacubėnas [20], Cherny and Shiryaev [6], as well as in Selivanov [26]. The proof contained in the first paper is inspired by the work of Eberlein and Jacod [12] and the proof in the other papers more or less use the idea of the Esscher transform (as we do here); the proofs sometimes are slightly more

[1]In view of remarks and questions that arose during presentations of the material stemming from the author's Ph.D thesis [22], it became clear that there was a desire for a self-contained treatment of the FTAP for exponential Lévy models. In this respect, one of the main results (Theorem 2.7) is *dedicated* to those who expressed interest for it, with the hope that it will help their teaching.

complicated and — as already mentioned — are valid in a one-dimensional, unconstrained setting.

It seems reasonable to proceed in proving analogous no-free-lunch equivalences in the case we have convex, but *not necessarily conic* constraints. The moment that we try to do so, we face an unexpected barrier: *no-free-lunch conditions are no longer sufficient to provide us with an equivalent supermartingale measure* — this slightly surprising fact in illustrated in Example 3.1. In the quest for finding an appropriate version of the FTAP under convex constraints we have to depart from the world of equivalent supermartingale measures and enter the realm of *equivalent supermartingale deflators*, i.e., state-price-density processes that are only supermartingales (and not martingales) and can therefore *lose mass*. A particularly efficient way to obtain an equivalent supermartingale deflator is by use of the numéraire portfolio: this is a special strategy that generates a wealth process in such a way that relative wealth processes generated by all other portfolios with respect to it are supermartingales. When the numéraire portfolio exists, so do equivalent supermartingale deflators — the interesting fact is that the converse also holds: existence of at least one equivalent supermartingale deflator will imply that the numéraire portfolio exists. This also turns out to be equivalent to requiring that the terminal values of all wealth processes that start from unit capital are bounded in probability, and it is exactly that last no-free-lunch notion (which we baptize *no unbounded profit with bounded risk*) that is tailor-made for the case of convex constraints in order to obtain an equivalent of the FTAP. We state this result as Theorem 3.5; its proof is more technical than that of Theorem 2.7 and its backbone is Lemma 4.1, whose proof is the whole the purpose of section 4. To the best of the author's knowledge, *no* result of this type (dealing with convex but not necessarily conic constraints) has appeared before in the literature. We note that the decision to single out the statement and proof of Lemma 4.1 is made not only for presentation reasons; it will also be used in a crucial way in Karatzas and Kardaras [21], where a study of the general semimartingale case is made. We further note that a solution to the problem of maximizing expected log-utility (which is very closely related to the numéraire portfolio, as is also discussed in subsection 3.8) in a general semimartingale model and under convex constraints has been carried by Goll and Kallsen [17].

Let us mention two more results that appear in the text. First, *completeness* for multi-asset exponential Lévy models is considered in subsection 2.6 — because it is not the main point of this paper, the treatment is very brief. Second, a result concerning the infinite-time horizon case is given — Theorem 3.7. If existence of free lunches is the *exception* when dealing with finite-time planning horizon, since it happens in the most severe way, it is the *rule* in infinite-time horizon models: one is *always* able to construct a free lunch, provided that the *original* probability is not a supermartingale measure. In the one-dimensional case, a statement of this last result appears and is proved in Selivanov [26]; nevertheless, it is not clear how to transfer the statement appearing there to the multi-dimensional case that we are dealing here.

0.2. **Organization of the paper.** This Introduction continues with fixing notation and discussing basic facts concerning Lévy processes. Section 1 introduces the financial market with exponential Lévy discounted stock-price processes and describes wealth processes as well as constraints. In section 2 we introduce the no-free-lunch and equivalent-(super)martingale notions that shall be used in the sequel and we present the first main result: Theorem 2.7, that provides equivalences for the cone-constrained case. We proceed in section 3 to introduce the numéraire portfolio, equivalent supermartingale deflators; then, we state Theorem 3.5 that covers no-free-lunch equivalences for cone-constrained models. In the same section we provide a result concerning the infinite-time horizon case (Theorem 3.7) and discuss the connection between the numéraire portfolio and the growth-optimal, that actually give us the way to *construct* it. Section 4 contains only the statement and proof of Lemma 4.1 that is needed to complete the proof of Theorem 3.5. We also include an Appendix with some special results on Lévy processes that are needed in the main text.

0.3. **Some notation.** The transpose of a vector $x \in \mathbb{R}^d$ is denoted by $x^\top$, its *norm* is $|x| := \sqrt{x^\top x}$, and superscripts denote coordinates: $x = (x^1, \dots x^d)^\top$. The *indicator function* of a set A is denoted by $\mathbb{I}_A$; for subsets of $\mathbb{R}^d$, we write $\{|x| > 1\}$ to actually express $\{x \in \mathbb{R}^d \,|\, |x| > 1\}$.

We are working on a *stochastic basis* $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where the *filtration* $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is right-continuous and augmented by all $\mathbb{P}$-null sets. The symbol $\mathbb{E}$ always denotes *expectation* of random variables under $\mathbb{P}$. Expectations with respect to other probability measures (say, $\mathbb{Q}$) will involve the measure appearing as superscript on $\mathbb{E}$ (say, $\mathbb{E}^{\mathbb{Q}}$).

For a d-dimensional semimartingale X and a d-dimensional predictable[2] process π, we shall denote by $\pi \cdot X$ the *vector* stochastic integral process whenever this makes sense, i.e., when π is *X-integrable*. One can check for example Jacod and Shiryaev [11] for these notions.

Any càdlàg (adapted, right-continuous with left-hand limits) process Z has an obviously-defined left-continuous — thus predictable — version Z_-; for concreteness, we set $Z_-(0) = 0$. We also define the *jump process* $\Delta Z := Z - Z_-$.

Finally, for a one-dimensional semimartingale Y, $\mathcal{E}(Y)$ will denote the *stochastic exponential* of Y, i.e., *unique* semimartingale Z that solves the stochastic differential equation $\mathrm{d}Z_t = Z_{t-}\mathrm{d}Y_t$.

0.4. **Basics of Lévy processes.** The are several good books that one can obtain information on Lévy processes — for example, Sato [25] is a good reference for the theoretical part, while Cont and Tankov [7] provide applications in financial modeling.

Given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, a d-dimensional càdlàg process L with $L_0 = 0$, such that for all $0 \leq s < t$, the increment $L_t - L_s$ is independent of $\mathcal{F}_s$ and its distribution only depends on $t - s$ will be called an **F**-*Lévy process*.

[2]The *predictable σ-algebra* is generated by all the adapted, left-continuous processes.

With a Lévy process L comes its *Lévy triplet* (b_L, c_L, ν_L). Here, $b_L \in \mathbb{R}^d$, c_L is a nonnegative-definite $d \times d$ matrix (if $d = 1$ this just reads $c_L \in \mathbb{R}_+$), and ν_L is a *Lévy measure* on $\mathbb{R}^d$ with its Borel σ-algebra, i.e., ν_L satisfies $\nu_L(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu_L(\mathrm{d}x) < +\infty$ (the wedge "$\wedge$" denotes minimum: $f \wedge g = \min\{f, g\}$). The finite-dimensional distributions of L are completely determined by its Lévy triplet via the characteristic functions

$$\mathbb{E}\exp\Big(i\sum_{j=1}^{n} u_j^\top (L_{t_j} - L_{t_{j-1}})\Big) = \prod_{j=1}^{n} \exp\big((t_j - t_{j-1})\phi(u_j)\big), \tag{0.1}$$

for all $0 = t_0 < \ldots < t_n$ and $u_j \in \mathbb{R}^d$ for all $j = 1, \ldots, n$, where $i = \sqrt{-1}$ and

$$\phi(u) := iu^\top b_L - \frac{u^\top c_L u}{2} + \int_{\mathbb{R}^d} (e^{iu^\top x} - 1 - iu^\top x \mathbb{I}_{\{|x| \le 1\}})\nu_L(\mathrm{d}x) \tag{0.2}$$

We have $\mathbb{E}|L_t| < \infty$ for all $t \in \mathbb{R}_+$ if and only if $\int_{\mathbb{R}^d} |x|\mathbb{I}_{\{|x|>1\}}\nu_L(\mathrm{d}x) < \infty$; then

$$\mathbb{E}L_t = t\Big(b_L + \int_{\mathbb{R}^d} x\mathbb{I}_{\{|x|>1\}}\nu_L(\mathrm{d}x)\Big). \tag{0.3}$$

In the one-dimensional case $d = 1$, formally setting $u = -i$ in (0.1) and (0.2) one obtains the *exponential formula* (written in logarithmic form to ease reading):

$$\log\big(\mathbb{E}e^{L_t}\big) = t\Big(b_L + \frac{c_L}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbb{I}_{\{|x| \le 1\}})\nu_L(\mathrm{d}x)\Big); \tag{0.4}$$

this always holds, in the sense that one side is finite if and only if the other is, and when they are finite they give the same value.

Further results on Lévy processes that will be useful later are collected in the Appendix.

1. Exponential Lévy Models of Financial Markets

1.1. **The financial market model.** The prices of d financial assets are modeled as d strictly positive semimartingales $\widetilde{S}^1, \ldots, \widetilde{S}^d$. There is also another process $\widetilde{S}^0$ which models the *money market* and plays the role of a "benchmark", in the sense that wealth processes will be quoted in units of $\widetilde{S}^0$. We then define the *discounted price processes* $S^i := \widetilde{S}^i / \widetilde{S}^0$ for $i = 0, \ldots, d$. The d-dimensional vector process $(S^1, \ldots, S^d)$ will be denoted by S.

We now enforce more structure on each of the discounted price-processes; in particular, we assume that they satisfy $\mathrm{d}S^i_t = S^i_{t-}\mathrm{d}X^i_t$, or equivalently $S^i = S^i_0\mathcal{E}(X^i)$, where for all $i = 1, \ldots, d$, X^i is a Lévy process with $\Delta X^i > -1$ (remember that $\mathcal{E}$ is the stochastic exponential operator). Denote by X the d-dimensional Lévy process $(X^1, \ldots, X^d)$. According to the *Lévy-Itô path decomposition* one can write

$$X_t = bt + \sigma\beta_t + \int_0^t \int_{\mathbb{R}^d} x\mathbb{I}_{\{|x| \le 1\}}\big(\mu(\mathrm{d}x, \mathrm{d}u) - \nu(\mathrm{d}x)\mathrm{d}u\big) + \int_0^t \int_{\mathbb{R}^d} x\mathbb{I}_{\{|x|>1\}}\mu(\mathrm{d}x, \mathrm{d}u). \tag{1.1}$$

With $c := \sigma\sigma^\top$, (b, c, ν) is the Lévy triplet of X. Here, β is a standard d-dimensional Brownian motion, and μ is the *jump measure* of X, i.e., the random counting measure defined for $t \in \mathbb{R}_+$ and $A \subseteq \mathbb{R}^d \setminus \{0\}$ by $\mu([0, t] \times A) := \sum_{0 \le s \le t} \mathbb{I}_A(\Delta X_s)$.

Since $S^i = S^i_0 \mathcal{E}(X^i)$, one can actually write $L^i := \log S^i$ in terms of X^i as follows: $L^i_t = L^i_0 + X^i_t - c^{ii}t/2 - \int_0^t \int_{\mathbb{R}^d} [x - \log(1+x)] \mu(\mathrm{d}x, \mathrm{d}u)$; we then observe that L^i is a Lévy process, and this is the reason why models like the ones we are considering are called *exponential Lévy models*. Both the usual and the stochastic logarithm of the asset prices are Lévy processes; we choose to state everything in terms of the stochastic — as opposed to the usual — logarithm since it will be much more convenient in the sequel.

We shall be mostly working on a finite-time horizon; only one result (Theorem 3.7) will be stated for the infinite-time horizon case. We then fix a number $T \in \mathbb{R}_+$ (the *maturity*) and we denote $[\![0, T]\!] := \Omega \times [0, T]$.

1.2. **Portfolios, wealth processes and constraints.** A financial agent starts with some strictly positive initial capital which we normalize to be unit throughout, and can invest in the assets by choosing a predictable, d-dimensional and X-integrable process process π, which we shall refer to as *portfolio*. We interpret π^i_t as the *proportion of current wealth* invested in stock i at time t; the remaining proportion of wealth $\pi_0 := 1 - \sum_{i=1}^d \pi^i$ is then invested in the money market. The wealth generated by this portfolio is constrained to remain strictly positive at all times; going on the red is not allowed in our model.

If W^π denotes the *discounted* wealth process obtained following π, then $W^\pi > 0$ and thus $\Delta W^\pi_t > -W^\pi_{t-}$. The previous interpretation for π implies that

$$\frac{\mathrm{d}W^\pi_t}{W^\pi_{t-}} \; = \; \sum_{i=0}^d \pi^i_t \frac{\mathrm{d}S^i_t}{S^i_{t-}} \; = \; \sum_{i=1}^d \pi^i_t \mathrm{d}X^i_t \; \equiv \; \pi^\top_t \mathrm{d}X_t, \tag{1.2}$$

the second equality simply holding because $\mathrm{d}S^0_t = 0$ and $\mathrm{d}S^i_t = S^i_{t-}\mathrm{d}X^i_t$.

The financial agent might be constrained further in the use of any desired portfolio position; we model this by introducing a *closed* and *convex* set $\mathfrak{C} \subseteq \mathbb{R}^d$ and requiring that $\pi(\omega, t) \in \mathfrak{C}$ for all $(\omega, t) \in [\![0, T]\!]$. For example, if the agent is prevented from selling stock short, we have $\mathfrak{C} = (\mathbb{R}_+)^d$. If we further prevent borrowing from the bank then we must also have $\pi^0 \geq 0$; in other words we must use $\mathfrak{C} = \{\mathrm{p} \in \mathbb{R}^d \,|\, \mathrm{p}^i \geq 0 \text{ and } \sum_{i=1}^d \mathrm{p}^i \leq 1\}$.

The constrains set $\mathfrak{C}$ should be such that we at least give freedom *not* to invest in the stock market if the agent chooses to do so. This should be modeled by requiring $0 \in \mathfrak{C}$, but there might also be *degeneracy* in the market, i.e., linear dependence of the returns of the stocks. The effect of this is that different portfolios will produce the same wealth. To understand how this notion should be formalized, consider two portfolios π_1 and π_2 with $W^{\pi_1} = W^{\pi_2}$. Uniqueness of the stochastic exponential implies $\pi_1 \cdot X = \pi_2 \cdot X$, or that $\zeta := \pi_2 - \pi_1$ will satisfy $\zeta \cdot X \equiv 0$, which is easily seen to be equivalent to $\zeta \cdot \beta = 0$, $\zeta^\top \Delta X = 0$ and $\zeta^\top b = 0$.

Definition 1.1. For a Lévy triplet (b, c, ν), the linear subspace of *null investments* $\mathfrak{N}$ is defined as the set of vectors $\mathfrak{N} := \big\{\zeta \in \mathbb{R}^d \,|\, \zeta^\top c = 0, \; \nu[\zeta^\top x \neq 0] = 0 \text{ and } \zeta^\top b = 0\big\}$.

Finally, here comes the formal definition of our portfolio strategies.

Definition 1.2. Consider a convex and closed $\mathfrak{C} \subseteq \mathbb{R}^d$ such that $\mathfrak{N} \subseteq \mathfrak{C}$. The class $\Pi_{\mathfrak{C}}$ of all $\mathfrak{C}$*-constrained portfolios* is defined to consist of all predictable and X-integrable processes π such that $\pi^\top \Delta X > -1$ and $\pi(\omega, t) \in \mathfrak{C}$ for all $(\omega, t) \in [\![0, T]\!]$.

Remark 1.3. (On Natural Constraints) Observe that the positivity requirement for W^π implies $\pi^\top \Delta X \geq -1$; in terms of the Lévy measure ν this is equivalent to $\nu[\pi^\top x < -1] = 0$. In other words, the set $\mathfrak{C}_0 := \{\mathrm{p} \in \mathbb{R}^d \,|\, \nu[\mathrm{p}^\top x < -1] = 0\}$ present some *already* model-enforced constraints, *regardless* of any other constraints $\mathfrak{C}$ enforced to agents. Thus, insofar as $\mathfrak{C}_0 \subseteq \mathfrak{C}$, we are basically regarding this as a unconstrained case.

Even though we could in principle enrich the given constraints $\mathfrak{C}$ to include the natural ones by considering $\mathfrak{C} \cap \mathfrak{C}_0$ we shall *not* do so — we regard $\mathfrak{C}$ as "outside" constraints.

From the wealth dynamics (1.2) it follows that for all $\pi \in \Pi_{\mathfrak{C}}$ we have $W^\pi = \mathcal{E}(\pi \cdot X)$. Observe that any *constant* vector $\pi \in \mathfrak{C}$ with $\nu[\pi^\top x \leq -1] = 0$ can be considered as an element of $\Pi_{\mathfrak{C}}$ and that the wealth it generates is again an exponential Lévy process, because $\pi \cdot X = \pi^\top X$ is a Lévy process.

Remark 1.4. The assumption $\mathfrak{N} \subseteq \mathfrak{C}$ on the constraint set implies that $\mathfrak{C} = \mathfrak{C} + \mathfrak{N}$: indeed, for any $\pi \in \mathfrak{C}$ and $\zeta \in \mathfrak{N} \subseteq \mathfrak{C}$ we have that $n\zeta \in \mathfrak{C}$ for any $n \in \mathbb{N}$, thus the convex combination $(1 - n^{-1})\pi + \zeta$ belongs to $\mathfrak{C}$ as well; since $\mathfrak{C}$ is closed, $\pi + \zeta \in \mathfrak{C}$. Now, $\mathfrak{C}$ is closed and $\mathfrak{N}$ is a linear subspace; this means that $\mathsf{pr}_{\mathfrak{N}^\perp} \mathfrak{C} = \mathfrak{C} \cap \mathfrak{N}^\perp$ is also closed in the subspace $\mathfrak{N}^\perp$, where $\mathsf{pr}_{\mathfrak{N}^\perp}$ is the usual Euclidean projection on $\mathfrak{N}^\perp$, the orthogonal complement of $\mathfrak{N}$. We conclude that we can restrict attention to the set $\mathfrak{C} \cap \mathfrak{N}^\perp$ for the portfolios — any degeneracy originally present in the market disappears there.

2. No-Free-Lunch Equivalences for Convex-Cone-Constrained Models

2.1. **Classical free-lunch-type notions.**

We remind ourselves of some "no free lunch" conditions that will be matter of our study later on.

Definition 2.1. For the following three definitions we consider our financial model with $\mathfrak{C}$-constrained portfolio class $\Pi_{\mathfrak{C}}$.

(1) A portfolio $\pi \in \Pi_{\mathfrak{C}}$ *generates an arbitrage*, if $\mathbb{P}[W_T^\pi \geq 1] = 1$ and $\mathbb{P}[W_T^\pi > 1] > 0$. If no such portfolio exists we say that the $\mathfrak{C}$-constrained market satisfies *no arbitrage* ($\mathrm{NA}_{\mathfrak{C}}$).

(2) The $\mathfrak{C}$-constrained market is said to satisfy the *no unbounded profit with bounded risk* ($\mathrm{NUPBR}_{\mathfrak{C}}$) condition if the collection of positive random variables $(W_T^\pi)_{\pi \in \Pi_{\mathfrak{C}}}$ is bounded in probability, i.e., if $\lim_{m \to \infty} \downarrow (\sup_{\pi \in \Pi_{\mathfrak{C}}} \mathbb{P}[W_T^\pi > m]) = 0$.

(3) A *free lunch with vanishing risk* is a sequence of portfolios $(\pi_n)_{n \in \mathbb{N}}$ with $\mathbb{P}[W_T^{\pi_n} \geq 1 - \delta_n] = 1$ for a decreasing sequence $\delta_n \downarrow 0$, such that there exists $\epsilon > 0$ with $\mathbb{P}[W_T^{\pi_n} > 1 + \epsilon] > \epsilon$. If such a situation is impossible by use of $\mathfrak{C}$-constrained portfolios, we say that the *no free lunch with vanishing risk* ($\mathrm{NFLVR}_{\mathfrak{C}}$) condition holds.

In the unconstrained case we skip the subscripts "$\mathbb{R}^d$" and write NA, NUPBR and NFLVR.

$\mathrm{NA}_{\mathfrak{C}}$ is the most classical of all three notions and its interpretation is straightforward. The $\mathrm{NUPBR}_{\mathfrak{C}}$ condition says that the probability of making "crazy" amounts of money at time T starting from unit capital and staying positive can be estimated uniformly over all portfolios and converges to zero as that "crazy" amount goes to infinity. $\mathrm{NFLVR}_{\mathfrak{C}}$ was introduced by Delbaen and Schachermayer [8] in order to prove a general version of the Fundamental Theorem of Asset Pricing. It can be further shown that if a free lunch with vanishing risk exists, we can choose $(W_T^{\pi_n})_{n \in \mathbb{N}}$ so that it converges $\mathbb{P}$-a.s. to a $[1, +\infty]$-valued random variable f which will (necessarily) satisfy $\mathbb{P}[f > 1] > 0$ — then, f is the free lunch and δ_n is the downside risk of using the portfolio π_n which *vanishes* to zero.

It is an easy exercise that $\mathrm{NFLVR}_{\mathfrak{C}}$ implies both $\mathrm{NA}_{\mathfrak{C}}$ and $\mathrm{NUPBR}_{\mathfrak{C}}$ and we shall use this fact later on — actually, $\mathrm{NFLVR}_{\mathfrak{C}} \Leftrightarrow \mathrm{NA}_{\mathfrak{C}} + \mathrm{NUPBR}_{\mathfrak{C}}$ if $\mathfrak{C}$ is a *cone* (see Karatzas and Kardaras [21]). In general semimartingale models, none of the two conditions $\mathrm{NA}_{\mathfrak{C}}$ and $\mathrm{NUPBR}_{\mathfrak{C}}$ implies the other, and they are not mutually exclusive; for exponential Lévy markets and cone constraints we shall see that they are equivalent.

2.2. **Unbounded Increasing Profit.** We now introduce yet another form of arbitrage — actually, the most egregious one: existence of wealth processes that start with unit capital, manage to make something, and are furthermore increasing.

Definition 2.2. Let $\check{\mathfrak{C}} := \bigcap_{a>0} a\mathfrak{C}$ be the *recession cone* of $\mathfrak{C}$. A $\pi \in \Pi_{\check{\mathfrak{C}}}$ is said to generate an *unbounded increasing profit* if W^π is increasing, i.e., if $\mathbb{P}[W_s^\pi \leq W_t^\pi, \forall 0 \leq s < t \leq T] = 1$, and if $\mathbb{P}[W_T^\pi > 1] > 0$. If no such portfolio exists we say that the *no unbounded increasing profit* ($\mathrm{NUIP}_{\mathfrak{C}}$) condition holds.

The process W^π is increasing if and only if $\pi \cdot X$ is increasing. The qualifier "unbounded" stems from the fact that since $\pi \in \Pi_{\check{\mathfrak{C}}}$, one can invest as much as one wishes on the strategy π; by doing so, the agent's wealth will be multiplied, and as the position becomes arbitrarily large, the gains are unbounded.

The $\mathrm{NUIP}_{\mathfrak{C}}$ condition is the weakest "no free lunch" notion of them all defined; both $\mathrm{NA}_{\mathfrak{C}}$ and $\mathrm{NUPBR}_{\mathfrak{C}}$ obviously imply it. Amazingly (or not so amazingly — see Lemma A.1) it turns out that in exponential Lévy markets and under cone constraints $\mathrm{NUIP}_{\mathfrak{C}}$ is equivalent to all previously-defined arbitrage notions. In other words, if any opportunities for free lunches exist in exponential Lévy models, they are of the most egregious type: unbounded increasing profits. Of course, the reason for this is the very special structure of exponential Lévy models that makes many "optimal" portfolios (for example, the ones that correspond to power utility functions) constant; this has been observed and known since the work of Foldes [15].

2.3. **Immediate arbitrage opportunities.** To obtain the connection of arbitrage — and especially the $\mathrm{NUIP}_{\mathfrak{C}}$ condition — with the Lévy triplet of X, we now give the definition of the immediate arbitrage opportunity vectors.

Definition 2.3. Let (b, c, ν) be any Lévy triplet. Define the set $\mathfrak{I}$ of *immediate arbitrage opportunities* to be the set of vectors $\xi \in \mathbb{R}^d \setminus \mathfrak{N}$ such that the following three conditions hold: (1) $\xi^\top c = 0$, (2) $\nu[\xi^\top x < 0] = 0$, and (3) $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \le 1\}} \nu(\mathrm{d}x) \ge 0$.

Observe that we are *not* considering null investments in the previous definition — a $\xi \in \mathfrak{N}$ satisfies the three conditions, but cannot be considered an "arbitrage opportunity" since it has zero returns. It is easy to see that $\mathfrak{I}$ is a cone with the whole "face" $\mathfrak{N}$ removed.

As Lemma 2.5 below will show, immediate arbitrage opportunities are constant portfolios that result in increasing profits. It is instructive to give examples in two special cases of Lévy processes, in order to also make comparison with previous work.

Example 2.4. We first consider the multi-dimensional Samuelson-Black-Scholes-Merton model, i.e., $X_t = bt + \sigma \beta_t$. Since $\nu \equiv 0$, an immediate arbitrage opportunity is a $\xi \in \mathbb{R}^d$ with $\xi^\top c = 0$ and $\xi^\top b > 0$. It then follows that absence of immediate arbitrage opportunities is equivalent to the existence of $\rho \in \mathbb{R}^d$ such that $b = c\rho$. The vector ρ always exists if c is nonsingular.

Consider now a general one-stock exponential Lévy model, which we assume to be nontrivial (in that $X \neq 0$; here this is equivalent to $\mathfrak{N} = \{0\}$). When do immediate arbitrage opportunities exist? Observe that if there exists a diffusion component, i.e., if $c > 0$, then $\mathfrak{I} = \emptyset$ because (1) of Definition 2.3 fails for all $\xi \neq 0$. If $c = 0$, then we only need to check (2) and (3) of Definition 2.3 for $\xi = 1$ and $\xi = -1$. Now, $\xi = 1$ is an immediate arbitrage opportunity if $\nu[x < 0] = 0$ and $b - \int x \mathbb{I}_{\{|x| \le 1\}} \nu(\mathrm{d}x) \ge 0$, and it is easy to see — or consult Lemma 2.5 to convince yourselves — that this is the case if and only if X (equivalently, the stock price S) is increasing. Similarly, $\xi = -1$ is an immediate arbitrage opportunity if and only if X, and equivalently S, is decreasing. We thus get exactly the condition that appears in [6] and [20].

The following lemma explains the relevance of the above Definition 2.3 with arbitrage.

Lemma 2.5. *Suppose that $\mathfrak{I} \neq \emptyset$. Then, $\xi \in \mathfrak{I}$ if and only if W^ξ is an increasing process and $\mathbb{P}[W^\xi_T > 1] > 0$. Thus, if further $\xi \in \check{\mathfrak{C}}$, then ξ is an unbounded increasing profit.*

Proof. Suppose that $\mathfrak{I} \neq \emptyset$ and pick $\xi \in \mathfrak{I}$. Condition (1) of Definition 2.3 implies that $\xi^\top \beta \equiv 0$ and condition (2) that $\pi^\top \Delta X \ge 0$; in particular, $\pi^\top X$ will then be a Lévy process of finite variation and we can write

$$\xi^\top X_t = t\Big(\xi^\top b - \int_{\mathbb{R}^d} \xi^\top x \mathbb{I}_{\{|x| \le 1\}} \nu(\mathrm{d}x)\Big) + \int_0^t \int_{\mathbb{R}^d} (\xi^\top x) \mu(\mathrm{d}x, \mathrm{d}t). \tag{2.1}$$

The last term $\int_0^t \int_{\mathbb{R}^d} (\xi^\top x) \mu(\mathrm{d}x, \mathrm{d}t)$ is a pure-jump increasing process, and since $\xi \in \mathfrak{I}$ we have $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \le 1\}} \nu(\mathrm{d}x) \ge 0$. Finally, since $\xi \notin \mathfrak{N}$ we must have that one of the two processes in the right-hand-side of (2.1) is nonzero; it follows that $\xi^\top X$ is increasing and nonzero, and thus $W^\xi = \mathcal{E}(\xi^\top X)$ is increasing and nonconstant ($\mathbb{P}[W^\xi_T > 1] > 0$).

Let us now assume that for some $\xi \in \mathbb{R}^d$ we have W^ξ being increasing; this is equivalent to saying that $\xi^\top X$ is increasing. But then it is of finite variation, thus $\xi^\top \beta = 0$, i.e.,

$\xi^\top c = 0$. Further, we must have $\xi^\top \Delta X \geq 0$ which is of course equivalent to $\nu[\xi^\top x < 0] = 0$. Finally, we can write $\xi^\top X$ as in (2.1) and since $\xi^\top X$ is increasing, the first term is continuous (linear) and the second pure-jump we must have $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x|\leq 1\}} \nu(\mathrm{d}x) \geq 0$. We have all three conditions of Definition 2.3, and finally if $\xi^\top X$ is nonzero we must have $\xi \notin \mathfrak{N}$, which gives $\xi \in \mathfrak{I}$. □

2.4. **Changes of measure that respect the exponential Lévy structure.** Absence of free lunches in the market is connected to existence of probability measures that are equivalent to the original and endow the stock price processes with some martingale-type property. In the context of exponential Lévy models it is actually possible to change the original probability $\mathbb{P}$ in such a way so that the exponential Lévy property remains intact. We now describe a way of doing so that will prove most useful in the proof of Theorem 2.7.

Pick $\eta \in \mathbb{R}^d$ and then some $g : \mathbb{R}^d \mapsto \mathbb{R}$ such that $g(x) = 0$ for $|x| \leq 1$, as well as $\int e^{-\eta^\top x - g(x)} \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) < +\infty$ — for example, this will hold for every $\eta \in \mathbb{R}^d$ if g is defined by $g(x) = 0$ for $|x| \leq 1$ and $g(x) = |x|^2 - 1$ for $|x| > 1$ (this is exactly the function g we shall use in the sequel). The process $Z^{(\eta,g)}$ defined by

$$Z_t^{(\eta,g)} := \exp\Big(- \eta^\top X_t - \sum_{0<s\leq t} g(\Delta X_s) - t\psi(\eta, g)\Big), \tag{2.2}$$

for some constant $\psi(\eta, g)$ is exponential Lévy and the exponential formula (0.4) give us that $\psi(\eta, g) := -\eta^\top b + \frac{1}{2}\eta^\top c \eta + \int (e^{-\eta^\top x - g(x)} - 1 + \eta^\top x \mathbb{I}_{\{|x|\leq 1\}}) \nu(\mathrm{d}x)$ makes $Z^{(\eta,g)}$ a martingale.

Define then a new probability measure $\mathbb{P}^{(\eta,g)}$ via $(\mathrm{d}\mathbb{P}^{(\eta,g)}/\mathrm{d}\mathbb{P})|_{\mathcal{F}_T} = Z_T^{(\eta,g)}$. Pick any positive Borel-measurable functional Φ that acts on cádlág processes and observe that for all $0 \leq t \leq T$ we have, with $\mathbb{E}^{(\eta,g)}$ denoting expectation under $\mathbb{P}^{(\eta,g)}$:

$$\begin{aligned}
\mathbb{E}^{(\eta,g)}\left[\Phi\big((X_{t+s} - X_t)_{0\leq s\leq T-t}\big) \;\big|\; \mathcal{F}_t\right] &= \mathbb{E}\Big[\frac{Z_T^{(\eta,g)}}{Z_t^{(\eta,g)}} \Phi\big((X_{t+s} - X_t)_{0\leq s\leq T-t}\big) \;\big|\; \mathcal{F}_t\Big] = \\
\mathbb{E}\big[\widehat{\Phi}\big((X_{t+s} - X_t)_{0\leq s\leq T-t}\big) \;\big|\; \mathcal{F}_t\big] &= \mathbb{E}\big[\widehat{\Phi}\big((X_s)_{0\leq s\leq T-t}\big)\big] = \\
\mathbb{E}[Z_{T-t}^{(\eta,g)} \Phi((X_s)_{0\leq s\leq T-t})] &= \mathbb{E}^{(\eta,g)}[\Phi((X_s)_{0\leq s\leq T-t})].
\end{aligned}$$

The functional $\widehat{\Phi}$ above has obvious definition. It follows that X is still a Lévy process under $\mathbb{P}^{(\eta,g)}$. Since $Z_t^{(\eta,g)} e^{iu^\top X_t} = \exp[(iu - \eta)^\top X_t - \sum_{0<s\leq t} g(\Delta X_s) - t\psi(\eta, g)]$, we have

$$\mathbb{E}^{(\eta,g)}[e^{iu^\top X_t}] = \exp\big(t(\psi(\eta - iu, g) - \psi(\eta, g))\big);$$

thus, the cumulant $\phi^{(\eta,g)}$ (the equivalent of (0.2) under the probability $\mathbb{P}^{(\eta,g)}$) satisfies $\phi^{(\eta,g)}(u) = \psi(\eta - iu, g) - \psi(\eta, g)$. Straightforward computations give the Lévy triplet $(b^{(\eta,g)}, c^{(\eta,g)}, \nu^{(\eta,g)})$ of X under $\mathbb{P}^{(\eta,g)}$ to be $b^{(\eta,g)} = b - c\eta + \int (e^{-\eta^\top x - g(x)} - 1) x \mathbb{I}_{\{|x|\leq 1\}} \nu(\mathrm{d}x)$, $c^{(\eta,g)} = c$ and $\nu^{(\eta,g)} = e^{-\eta^\top x - g(x)} \nu(\mathrm{d}x)$. Definition 2.3, coupled with the last equations involving the Lévy triplet of X under $\mathbb{P}^{(\eta,g)}$, imply that *the set $\mathfrak{I}$ of immediate arbitrage opportunities remains invariant* when we change from $\mathbb{P}$ to $\mathbb{P}^{(\eta,g)}$.

Let us finally remark that the transition from $\mathbb{P}$ to $\mathbb{P}^{(\eta,g)}$ can be carried out in two steps. First, we change $\mathbb{P}$ to $\mathbb{P}^{(0,g)}$ "lightening" the tails of the Lévy measure using the function e^{-g}, which turns out to be exactly the Radon-Nikodym derivative of $\nu^{(0,g)}$ (the Lévy measure of X under $\mathbb{P}^{(0,g)}$) with respect ν (the Lévy measure of X under $\mathbb{P}$). As a second step, we change $\mathbb{P}^{(0,g)}$ to $\mathbb{P}^{(\eta,g)}$, *exponentially tilting* $\mathbb{P}^{(0,g)}$. This exponential tilting method is also referred to as the *Esscher transform*.

2.5. **No-free-lunch equivalences for the cone-constrained case.** We are almost ready present a complete characterization of the arbitrage situation in exponential Lévy financial models for the finite time-horizon case and a constrained set $\mathfrak{C}$ that is a closed convex *cone* with $\mathfrak{N} \subseteq \mathfrak{C}$. There is one formal definition missing involving the ability to change the original measure $\mathbb{P}$ to some other equivalent probability measure $\mathbb{Q}$ such that the stock price process, or possibly only the allowed wealth processes W^π for $\pi \in \Pi_{\mathfrak{C}}$ have some kind of martingale property under $\mathbb{Q}$.

In the unconstrained case, the notion of an equivalent martingale measure (see Definition 2.6 below) does the trick for our no-free-lunch equivalences, but in the presence of constraints this is no longer the case. The reason is that free lunches are not allowed only for portfolios that take values in $\mathfrak{C}$. Further, we cannot even hope that all *wealth processes* are martingales. Take for example X to be the negative of a Poisson process and assume we are constrained in the cone of positive strategies $\mathfrak{C} = \mathbb{R}_+$. Under any measure $\mathbb{Q} \sim \mathbb{P}$, the process $S = \mathcal{E}(X)$ will be non-increasing and not identically equal zero, which prevents it from being (even a local) martingale. It is a supermartingale though, and this turns out to be the appropriate notion.

Definition 2.6. A probability $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ (we denote $\mathbb{Q} \sim \mathbb{P}$) will be called
• *equivalent martingale measure* (EMM in short) if the discounted stock-price S is a vector $\mathbb{Q}$-martingale.
• $\mathfrak{C}$*-constrained equivalent supermartingale measure* (ESMM$_{\mathfrak{C}}$ in short) if the wealth process W^π is a $\mathbb{Q}$-supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$. The class of all ESMM$_{\mathfrak{C}}$ is denoted by $\mathfrak{Q}_{\mathfrak{C}}$.

Stochastic integrals of martingales that are further positive processes are local martingales; this has been shown by Ansel and Stricker [2]. Further, it is well-known that positive local martingales are supermartingales. Thus, we get that an EMM a fortiori is an ESMM$_{\mathfrak{C}}$ for any $\mathfrak{C}$; of course the opposite does not hold in general.

Even if $\mathfrak{C}$ is just a convex set, it is easy to see that if an ESMM$_{\mathfrak{C}}$ exists then it is automatically an equivalent supermartingale measure for the market with *cone* constraints $\overline{\mathsf{cone}}(\mathfrak{C})$, the *closure of the smallest cone that contains* $\mathfrak{C}$; the proof of this simple statement is left to the diligent reader. Thus, if we want to prove any theorem concerning equivalent supermartingale measures we might as well assume cone constraints — the pure convex case is treated in the next section.

For exponential Lévy models, and even under the *weakest* of no-free-lunch conditions (namely, $\mathrm{NUIP}_{\mathfrak{C}}$), not only can we find an $\mathrm{ESMM}_{\mathfrak{C}}$, but we can do so in a matter that respects the exponential Lévy structure as was described in the previous subsection.

Theorem 2.7. *For an exponential Lévy model with closed convex cone constraints $\mathfrak{C}$ on a finite financial planning horizon $[0, T]$, the following are equivalent:*

(1) *There exists a $\mathbb{Q} \sim \mathbb{P}$ under which X remains a Lévy process and $\pi^\top X$ is a Lévy supermartingale for all $\pi \in \mathfrak{C}$.*
(2) *The* $\mathrm{ESMM}_{\mathfrak{C}}$ *condition holds: $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$;*
(3) *The* $\mathrm{NFLVR}_{\mathfrak{C}}$ *condition holds;*
(4) *The* $\mathrm{NA}_{\mathfrak{C}}$ *condition holds;*
(5) *The* $\mathrm{NUPBR}_{\mathfrak{C}}$ *condition holds;*
(6) *The* $\mathrm{NUIP}_{\mathfrak{C}}$ *condition holds;*
(7) $\mathfrak{I} \cap \mathfrak{C} = \emptyset$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious: (1) is stronger than (2).

For (2) $\Rightarrow$ (3), we have that W^π for all $\pi \in \Pi_{\mathfrak{C}}$ is a positive $\mathbb{Q}$-supermartingale. Consider a sequence $(\pi_n)_{n\in\mathbb{N}}$ of elements in $\Pi_{\mathfrak{C}}$ that is a candidate for being a free lunch with vanishing risk, i.e., suppose that there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \downarrow 0$ and $\mathbb{P}[W_T^{\pi_n} \geq 1 - \delta_n] = 1$. Then, for all $\epsilon > 0$, $(1+\epsilon)\mathbb{Q}[W_T^{\pi_n} > 1+\epsilon] + (1-\delta_n)(1 - \mathbb{Q}[W_T^{\pi_n} > 1+\epsilon]) \leq \mathbb{E}^{\mathbb{Q}} W_T^{\pi_n} \leq 1$, which by simple algebra manipulations implies $\mathbb{Q}[W_T^{\pi_n} > 1+\epsilon] \leq \delta_n/(\epsilon+\delta_n)$. The right-hand-side of this last inequality converges to zero as n tends to infinity; since $\mathbb{P} \sim \mathbb{Q}$ we have that $\lim_{n\to\infty} \mathbb{P}[W_T^{\pi_n} > 1+\epsilon] = 0$ as well, and NFLVR holds.

The implications (3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are an easy exercise (use Definition 2.1), and implications (4) $\Rightarrow$ (6) and (5) $\Rightarrow$ (6) are even easier.

Implication (6) $\Rightarrow$ (7) is one direction of Lemma 2.5.

The cycle will be closed as soon as we prove (7) $\Rightarrow$ (1), which is the harder one. As mentioned in the Introduction, we follow the idea of Rogers [24], who applied it for discrete-time processes. Using the notation of the previous subsection 2.4, begin by changing the measure $\mathbb{P}$ into $\mathbb{P}^{(0,g)}$, where g is defined by $g(x) = 0$ for $|x| \leq 1$ and $g(x) = |x|^2 - 1$ for $|x| > 1$. Then $\mathbb{E}^{(0,g)}[\exp(|X_T|^2)] < \infty$; this is due to the behavior of the tails of the Lévy measure $\nu^{(0,g)}$ (in the notation of subsection 2.4) under $\mathbb{P}^{(0,g)}$ — one can check for example Sato [25] for matters like this. Since X is still a Lévy process under $\mathbb{P}^{(0,g)}$ and $\mathfrak{I}$ remains invariant under this change of measure we might as well assume from the outset that $\mathbb{E}[\exp(|X_T|^2)] < \infty$ (i.e., $\mathbb{P} \equiv \mathbb{P}^{(0,g)}$).

We proceed by considering the *exponential utility* function $U(x) := 1 - e^{-x}$ and setting $\phi(\mathrm{p}) := \mathbb{E} U(\mathrm{p}^\top X_T) = 1 - \mathbb{E}[e^{-\mathrm{p}^\top X_T}]$. The function ϕ is real-valued (because $\mathbb{E}\exp(|X_T|^2) < \infty$) and concave. Let $\phi_* := \sup_{\mathrm{p}\in\mathfrak{C}} \phi(\mathrm{p})$; since $\phi(\mathrm{p}) = \phi(\mathrm{p}+\zeta)$ for $\zeta \in \mathfrak{N}$, nothing changes if we restrict this infimum on $\mathfrak{N}^\perp$ (see Remark 1.4). Clearly, $\phi_* \geq \phi(0) = 0$.

We claim that if $\mathfrak{I} = \emptyset$, the supremum ϕ_* is achieved by a point in $\mathfrak{N}^\perp \cap \mathfrak{C}$. Otherwise, there would exist a sequence $(\mathrm{p}_n)_{n\in\mathbb{N}}$ in $\mathfrak{N}^\perp \cap \mathfrak{C}$ such that $\lim_{n\to\infty} \uparrow |\mathrm{p}_n| = +\infty$, $\phi(\mathrm{p}_n) \in$

$\mathbb{R}_+$ and $\lim_{n\to\infty} \phi(\mathrm{p}_n) = \phi_*$. Then, set $\xi_n := \mathrm{p}_n/|\mathrm{p}_n|$ and fix $a \in \mathbb{R}_+$; eventually, for all $n \geq n_a$ where n_a is large enough to satisfy $a \leq |\mathrm{p}_{n_a}|$, we have $a\xi_n \in \mathfrak{N}^\perp \cap \mathfrak{C}$ and $\phi(a\xi_n) \geq 0$ (the last follows from concavity of ϕ as soon as one remembers that $\phi(0) = 0$ and $\phi(\mathrm{p}_n) \geq 0$). Since $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of unit vectors in $\mathfrak{N}^\perp \cap \mathfrak{C}$ we can assume without loss of generality that it converges to some unit vector $\xi \in \mathfrak{N}^\perp \cap \mathfrak{C}$ (choosing a subsequence otherwise). Since $U(x) \leq 1$ for all $x \in \mathbb{R}$, Fatou's lemma is applicable and will give

$$\phi(a\xi) = \mathbb{E}U(a\xi^\top X_T) \geq \limsup_{n\to\infty} \mathbb{E}U(a\xi_n^\top X_T) = \limsup_{n\to\infty} \phi(a\xi_n) \geq 0.$$

In other words, $\mathbb{E}[(e^{-\xi^\top X_T})^a] \leq 1$ for all $a \in \mathbb{R}_+$; this can only hold if $\mathbb{P}[\xi^\top X_T \geq 0] = 1$. Since $\xi^\top X$ is a Lévy process, Lemma A.1 suggests that $\xi^\top X$ is increasing; since $\xi \in \mathfrak{N}^\perp$, Lemma 2.5 would finally give $\xi \in \mathfrak{I} \cap \mathfrak{C}$, which is assumed empty. We reached a contradiction to our assumption because we assumed that the supremum of ϕ is not attained by any vector in $\mathfrak{N}^\perp \cap \mathfrak{C}$. Thus, there exists $\eta \in \mathfrak{N}^\perp \cap \mathfrak{C}$ such that $\phi(\eta) = \phi_*$.

Now, pick any $\mathrm{p} \in \mathfrak{C}$ and observe that $\mathbb{R}_+ \ni a \mapsto \phi(\eta + a\mathrm{p})$ is concave in $a \in \mathbb{R}_+$ that has a maximum at $a = 0$. It follows that

$$\mathbb{E}\Big[\frac{e^{-\eta^\top X_T} - e^{-(\eta+a\mathrm{p})^\top X_T}}{a}\Big] = \frac{\phi(\eta + a\mathrm{p}) - \phi(\eta)}{a} \leq 0, \text{ for all } a > 0.$$

The concavity of $x \mapsto e^{-x}$ implies that the expression inside the expectation above is an increasing function of decreasing a; it is also clear that it converges $\mathbb{P}$-a.s. to $e^{-\eta^\top X_T}\mathrm{p}^\top X_T$ as $a \downarrow 0$. Since ϕ is finite-valued, we can use the monotone convergence theorem to get $\mathbb{E}[e^{-\eta^\top X_T}\mathrm{p}^\top X_T] \leq 0$. In other words, defining $\mathbb{P}^{(\eta,0)}$ as in subsection 2.4 we get $\mathbb{E}^{(\eta,0)}[\mathrm{p}^\top X_T] \leq 0$ for all $\mathrm{p} \in \mathfrak{C}$. This means that $\mathrm{p}^\top X$ is a Lévy supermartingale for all $\mathrm{p} \in \mathfrak{C}$. □

Remark 2.8. (On the Unconstrained Case) Recall from Remark 1.3 the natural constraints set $\mathfrak{C}_0$. Then If $\mathfrak{C}_0 \subseteq \mathfrak{C}$, i.e., in the unconstrained case, then one can replace conditions (1) and (2) of Theorem 2.7 above by

(1') *There exists* $\mathbb{Q} \sim \mathbb{P}$ *under which* X *is Lévy martingale and* S *martingale.*

(2') *An* EMM *exists*;

Indeed (1') $\Rightarrow$ (2') is obvious, while (1) $\Rightarrow$ (1') follows like this: $\mathrm{p}^\top X$ being a $\mathbb{Q}$-martingale for all $\mathrm{p} \in \mathbb{R}^d$ means that X is a $\mathbb{Q}$-martingale. Then, each S^i, $i = 1, \ldots, d$ is a positive local martingale; the exponential formula (0.4) gives $\mathbb{E}^{\mathbb{Q}} S_T = S_0$, i.e., that S is a martingale.

Remark 2.9. (Martingale vs σ-Martingale Measures) In their seminar work, Delbaen and Schachermayer [10] have showed that in a general semimartingale model in the unconstrained case and a possibly non locally bounded asset-price process S, the NFLVR condition is equivalent to existence of some $\mathbb{Q} \sim \mathbb{P}$ such that S is a *σ-martingale* under $\mathbb{Q}$ (which basically means that we can write S as a stochastic integral of a martingale).

For exponential Lévy markets, it turns out from the previous remark that *any* of our no-free-lunch conditions is equivalent to the existence of an EMM. There has been work

from some authors (we mention for example Cherny [5] and Yan [27]) on establishing a version of the FTAP in which no-free-lunch criteria are equivalent to the existence of an EMM, instead of simple a σ-martingale one. Obviously, these no-free-lunch criteria are equivalent to the ones mentioned in Theorem 2.7. In particular, Yan's work [27] allows us to conclude that we can *enlarge* the class of strategies that agents can use. Indeed, any predictable process θ (where now θ^i_t is perceived as the *units* of asset i that is held by the agent at time t) such that $\theta \cdot S \geq -a(1 + \sum_{i=1}^d S^i)$ for some $a > 0$ is allowed, and will not lead to free lunch.

Remark 2.10. (On Exponential Utility Maximization) The $\mathrm{ESMM}_{\mathfrak{C}}$ $\mathbb{Q}$ in the proof of equivalence (7) $\Rightarrow$ (1) in Theorem 2.7 above is constructed via *exponential utility maximization* in the financial market where the "original" probability measure is $\mathbb{P}^{(0,g)}$. We are not able to use directly $\mathbb{P}$ because $\mathbb{E}[e^{\mathrm{p}^\top X_T}]$ might be infinite for some $\mathrm{p} \in \mathfrak{C}$; in case $\mathbb{E}[e^{\mathrm{p}^\top X_T}] < \infty$ for all $\mathrm{p} \in \mathfrak{C}$ we can proceed with the proof and the measure $\mathbb{Q} = \mathbb{P}^{(\eta,0)}$ that we end up with is the *minimal entropy martingale measure*. The theme has received a lot of attention, let us just mention here that it has been treated by Fujiwara and Miyahara [16] and recently by Esche and Schweizer [14], as well as Hubalek and Sgarra [18].

Nevertheless, if $\mathbb{E}[e^{\mathrm{p}^\top X_T}]$ could take possibly infinite values, things are slightly more complicated. In that case, we can still find a vector $\eta \in \mathfrak{C}$ such that $\mathbb{E}[U(\eta^\top X_T)] \geq \mathbb{E}[U(\mathrm{p}^\top X_T)]$ for all $\mathrm{p} \in \mathfrak{C}$ (under the assumption $\mathfrak{I} = \emptyset$, of course), but *we cannot conclude that* $\mathbb{P}^{(\eta,0)}$ *is an equivalent martingale measure*. Take for example a unconstrained, one-stock exponential Lévy model with $c = 0$ and Lévy measure ν of the form $\nu(\mathrm{d}x) = f(x)\mathrm{d}x$ with $f(x) > 0$ for all $x \geq 1$ (so that $\mathfrak{I} = \emptyset$), and (i) $\int e^{ax}\mathbb{I}_{\{x>1\}} f(x)\mathrm{d}x = \infty$ for all $a > 0$, (ii) $\int x\mathbb{I}_{\{x>1\}} f(x)\mathrm{d}x < \infty$, and (iii) $b + \int x\mathbb{I}_{\{x>1\}} f(x)\mathrm{d}x < 0$. An example of such density f satisfies $f(x) \sim x^{-p}$ as $x \to \infty$ for some $p > 2$; then (i) and (ii) hold automatically and an appropriate choice of small enough b will ensure (iii) as well. Now, with $\phi(\mathrm{p}) := 1 - \mathbb{E}e^{-\mathrm{p}X_T}$ we have $\phi(\mathrm{p}) = -\infty$ for all $\mathrm{p} < 0$, and a simple use of Jensen's inequality gives $\phi(\mathrm{p}) < 0 = \phi(0)$ for all $\mathrm{p} > 0$ (because by (iii) we have $\mathbb{E}[\mathrm{p}X_T] < 0$ for $\mathrm{p} > 0$). It follows that the optimal portfolio is $\eta = 0$; this gives us $\mathbb{Q} = \mathbb{P}$, which is *not* an equivalent martingale measure, since $\mathbb{E}X_T < 0$ by (iii). Observe nevertheless that it is an ESMM, and it can be shown that it will always be — this is not just a coincidence here.

2.6. **Completeness.** Though not our main concern, we give here a characterization of *completeness* (the ability to perfectly replicate any bounded contingent claim) in exponential Lévy markets. We do not provide full details — we trust they can be filled by the reader. We note however that the weak martingale representation property for the filtration generated by a Lévy process as described for example in Jacod and Shiryaev [11] will have to be used.

Definition 2.11. The exponential Lévy market in a finite time-horizon $[0, T]$ is called *complete* if for all positive and bounded $H \in \mathcal{F}_T$ one can find $\pi \in \Pi$ and $x > 0$ such that $xW^\pi_T = H$.

In order to talk about completeness one should better assume that we are in the unconstrained case $\mathfrak{C} = \mathbb{R}^d$ (thus the absence of a subscript from Π in the definition above), and that *the filtration* $\mathbf{F}$ *is the usual augmentation of the one generated by* S, or equivalently of the one generated by X. These conditions are in force for this subsection.

We decompose $\mathbb{R}^d = \mathfrak{K} \oplus \mathfrak{K}^\perp$, where $\mathfrak{K} := \{x \in \mathbb{R}^d \mid cx = 0\}$ is the *kernel* of the covariance matrix c and $\mathfrak{K}^\perp$ is its orthogonal complement, and we denote by k the dimension of the linear subspace $\mathfrak{K}$. We also denote by $\mathsf{supp}(\nu)$ the *support* of the measure ν, i.e., the smallest closed subset of $\mathbb{R}^d$ that ν gives full measure.

Proposition 2.12. *With the assumptions and notation set above (in particular, $\mathfrak{C} = \mathbb{R}^d$) and an exponential Lévy market on a finite time-horizon $[0, T]$, suppose that the model satisfies any (and thus all) of the equivalent conditions of Theorem 2.7. The following are equivalent:*

(1) *The exponential Lévy model is complete.*
(2) *There exists a* unique EMM $\mathbb{Q}$.
(3) *We have (i) $\mathsf{supp}(\nu) \subseteq \mathfrak{K}$, (ii) $\mathsf{supp}(\nu)$ contains at most k points.*

One can start directly from the exponential Lévy model and not assume that it satisfies the equivalent conditions of Theorem 2.7. In that case, (1) should be substituted with
(1') The exponential Lévy model satisfies any of the conditions of Theorem 2.7 *and* is complete.
Implication (2) remains the same, while for (3) we have to add an extra requirement (3 iii) appearing below. To prepare the ground, notice that if (3) holds, and with $X^{\mathfrak{K}}$ denoting the orthogonal projection of X on $\mathfrak{K}$, we have $X^{\mathfrak{K}}_t = at + \sum_{n=1}^{N_t} Y_n$, for $a \in \mathfrak{K}$, N a Poisson process with some arrival rate $\lambda > 0$, and $(Y_n)_{n \in \mathbb{N}}$ a sequence of i.i.d. (and independent of N) random variables with simple discrete distributions charging less than k points on $\mathfrak{K}$. Condition $\mathfrak{I} = \emptyset$ of Theorem 2.7 is now equivalent to the following:
(3 iii) if $\xi \in \mathfrak{K}$ satisfies $\xi^\top a \geq 0$ and $\xi^\top x \geq 0$ for all $x \in \mathsf{supp}(\nu)$, then we actually have $\xi^\top a = 0$ and $\xi^\top x = 0$ for all $x \in \mathsf{supp}(\nu)$.

3. The Numéraire Portfolio, Supermartingale Deflators and No-Free-Lunch Equivalences for Convex-Constrained Models

In this section we aim in extending the scope of Theorem 2.7 to the convex-constrained case. As a byproduct we shall obtain even more equivalences for the cone-constrained and unconstrained case then the ones covered by Theorem 2.7. We introduce a very special portfolio that will help us do that. As discussed in Remark 2.10, in the course of proving Theorem 2.7 we used the optimal portfolio for *exponential* utility for a possibly changed probability measure; vis-à-vis, here we shall use the optimal portfolio for *logarithmic* utility under the *original* measure $\mathbb{P}$. This will enable us to prove equivalences valid under closed and convex — but not necessarily cone — constraints; more importantly, it is exactly *this* result that allows for generalization in general semimartingale models. The drawback is that we have to work harder; part of the proof of the main result here (Theorem 3.5) is

more technical and long, and will be the focus of the next section — this contrasts the (fair) easiness of the proof of Theorem 2.7. After the work is done, we continue the story in Karatzas and Kardaras [21] for the semimartingale case.

3.1. **The inadequacy of equivalent supermartingale measures.** As soon as we face *non-conic* convex constraints, the $\mathrm{NA}_{\mathfrak{C}}$ — or even $\mathrm{NFLVR}_{\mathfrak{C}}$ — condition is not any more sufficient to imply existence of an equivalent supermartingale measure.

Example 3.1. We take X be a 2-dimensional compound Poisson process, i.e., $X_t = \sum_{i=1}^{N_t} Y_i$, for $t \in [0, T]$, where N is a standard Poisson process and Y_i is a sequence of 2-dimensional independent and identically distributed random variables with $Y_i = (e_i, f_i - 1)$, e_i and f_i being independent with a standard exponential distribution (we only use the fact that they are independent and their distributions are supported on the positive half-line — even less is needed as the reader will note). Of course, in the unconstrained case there is clear arbitrage: take a strict long position in the first stock and null position on the second. Consider now the constraints set $\mathfrak{C} := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 \leq y\}$, i.e., only points on and above the parabola $y = x^2$ are allowed for investing. We claim that $\mathrm{NFLVR}_{\mathfrak{C}}$ holds, but no $\mathrm{ESMM}_{\mathfrak{C}}$ exists.

To see that no $\mathrm{ESMM}_{\mathfrak{C}}$ exists is easy: we have already noted that if it did it should already be an equivalent supermartingale measure for the market with constrains $\overline{\mathsf{cone}}(\mathfrak{C}) = \mathbb{R}_+ \times \mathbb{R}$; the latter is clearly impossible, since there is arbitrage.

In the process of showing the no free lunches exist for the $\mathfrak{C}$-constrained market, we use the following observation: for $\mathrm{p} \equiv (x, y) \in \mathfrak{C} \setminus \{0\}$ it *must* be that $y > 0$ (due to the constraints $y = 0 \Rightarrow x = 0$); also, since $\mathbb{P}[e_1 > 0] = 1$, we have $x e_1 + y(f_1 - 1) \leq \sqrt{y} e_1 + y(f_1 - 1)$. Then, $\mathbb{P}[\mathrm{p}^\top \Delta Y_1 < 0] \geq \mathbb{P}[e_1 < \sqrt{y}(1 - f_1)] > 0$; this should already give you a hint why no $\mathfrak{C}$-constrained arbitrage exists.

We now show that $\mathrm{NA}_{\mathfrak{C}}$ holds. Pick any portfolio $\pi \in \Pi_{\mathfrak{C}}$ that is supposed to generate an arbitrage and define $\tau := \inf\{t \in [0, T] \,|\, \Delta W_t^\pi \neq 0\}$, where we set $\tau = T$ when the set that we are taking the infimum is empty. It is obvious that τ is an $\mathbf{F}$-stopping time; actually, with $\tau_n := \inf\{t \in \mathbb{R}_+ \,|\, N_t = n\}$ denoting the n^{th} jump of N, we have $\{\tau = \tau_n\} = \{\pi_{\tau_k} = 0 \text{ for all } k < n,\ \pi_{\tau_n} \neq 0\} \in \mathcal{F}_{\tau_n -}$, a fact that will be important. Now, $\{\tau = T\} \subseteq \{W_T^\pi = 1\}$, thus if $\mathbb{P}[\tau = T] = 1$ we have $\mathbb{P}[W_T^\pi = 1] = 1$ and π is not an arbitrage. Suppose then that $\mathbb{P}[\tau < T] > 0$; we shall show that $\mathbb{P}[W_T^\pi < 1,\ \tau < T] > 0$, and then $\mathrm{NA}_{\mathfrak{C}}$ readily follows. Define the *second* time that a wealth readjustment happens $\tau' := \inf\{t \in (\tau, T] \,|\, \Delta W_t^\pi \neq 0\}$, where again we set $\tau' = T$ if the last set is empty. τ' is an $\mathbf{F}$-stopping time and we have $\mathbb{P}[W_T^\pi < 1] \geq \mathbb{P}[W_T^\pi < 1,\ \tau < T,\ \tau' = T] = \mathbb{P}[\pi_\tau^\top \Delta X_\tau < 0,\ \tau < T,\ \tau' = T]$. Since $\{\tau = \tau_n\} \in \mathcal{F}_{\tau_n -}$, Lemmata A.2 and A.3 in the Appendix give that $\pi_\tau \in \mathcal{F}_{\tau -}$ is independent of ΔX_τ and that the latter jump is distributed as Y_1. On $\{\tau < T\}$ we have $\pi_\tau \in \mathfrak{C} \setminus \{0\}$; the observation made in the previous paragraph coupled with the trivial fact $\mathbb{P}[\tau < T,\ \tau' = T] > 0$ imply $\mathbb{P}[\pi_\tau^\top \Delta X_\tau < 0,\ \tau < T,\ \tau' = T] > 0$, and thus $\mathbb{P}[W_T^\pi < 1] > 0$. We conclude that $\mathrm{NA}_{\mathfrak{C}}$ holds for this constrained market.

The fact that $\mathrm{NA}_{\mathfrak{C}}$ holds implies that actually $\mathrm{NFLVR}_{\mathfrak{C}}$ holds as well. The reason is that finite-time-horizon compound-Poisson-process models are equivalent to discrete-time models with a stochastic, but *finite* time-horizon; for discrete-time models, it is not hard to see that $\mathrm{NFLVR}_{\mathfrak{C}}$ is equivalent to the generally weaker $\mathrm{NA}_{\mathfrak{C}}$ (this is no longer true for infinite time-horizon models).

3.2. **The numéraire portfolio.** The following concept will prove crucial.

Definition 3.2. A portfolio $\rho \in \Pi_{\mathfrak{C}}$ will be called *numéraire portfolio* for the class $\Pi_{\mathfrak{C}}$, if for every other $\pi \in \Pi_{\mathfrak{C}}$ the *relative wealth process* W^{π}/W^{ρ} is a supermartingale.

The reader is referred to in Becherer [3] for the definition and more on this concept. The numéraire portfolio has many optimality properties; you can check Karatzas and Kardaras [21], where an extensive discussion on the existence of the numéraire portfolio for general semimartingale models and its relationship with free lunches is taking place.

Example 3.3. The numéraire portfolio exists and is equal to zero if and only if all wealth processes W^{π} for $\pi \in \Pi_{\mathfrak{C}}$ are $\mathbb{P}$-supermartingales. This is a trivial example, but it will find use in Theorem 3.7 where arbitrage in infinite-time horizon exponential Lévy markets is studied.

3.3. **Equivalent supermartingale deflators.** We introduce a concept that is weaker — but very closely related — to equivalent supermartingale measures. Let us assume that the numéraire portfolio exists; by way of definition, the process $(W^{\rho})^{-1}$ acts as a "deflator", under which all wealth processes W^{π} for $\pi \in \Pi_{\mathfrak{C}}$ become supermartingales. There are more processes sharing this last property.

Definition 3.4. A process D will be called a *$\mathfrak{C}$-constrained equivalent supermartingale deflator* ($\mathrm{ESMD}_{\mathfrak{C}}$) if $D_0 = 1$, $D_T > 0$ and such that DW^{π} is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$. The class of all $\mathrm{ESMD}_{\mathfrak{C}}$'s is denoted by $\mathfrak{D}_{\mathfrak{C}}$.

A $\mathrm{ESMM}_{\mathfrak{C}}$ (say, $\mathbb{Q}$) generates an $\mathrm{ESMD}_{\mathfrak{C}}$ D via the density process $D_t = (\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})\,|_{\mathcal{F}_t}$, for $t \in [0,T]$, so that $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset \Rightarrow \mathfrak{D}_{\mathfrak{C}} \neq \emptyset$. The reverse implication $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset \Rightarrow \mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$ does not hold in general as a simple example involving the notorious three-dimensional Bessel process shows; see Delbaen and Schachermayer [9]. Nevertheless, for exponential Lévy models and under *cone* constraints we shall soon see that $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset \Rightarrow \mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$ *does* hold.

3.4. **The main result.** Here is the result that puts the numéraire portfolio in the context of arbitrage. The difficult implication below is (5) $\Rightarrow$ (1) and will be the result of discussion in the subsequent subsections and the following section 4.

Theorem 3.5. *For an exponential Lévy model under closed convex constraints $\mathfrak{C} \subseteq \mathbb{R}^d$ on a finite-time horizon $[0,T]$, the following are equivalent:*

(1) *The numéraire portfolio exists in the class $\Pi_{\mathfrak{C}}$.*

(2) *An* ESMD$_{\mathfrak{C}}$ *exists:* $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$.
(3) *The* NUPBR$_{\mathfrak{C}}$ *condition holds.*
(4) *The* NUIP$_{\mathfrak{C}}$ *condition holds.*
(5) $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

If $\mathfrak{C}$ *is further a cone* ($\mathfrak{C} = \check{\mathfrak{C}}$)*, (1) and (2) above are equivalent to all conditions of Theorem 2.7.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial: $(W^{\rho})^{-1}$ is an element of $\mathfrak{D}_{\mathfrak{C}}$.

Now, for the implication (2) $\Rightarrow$ (3), start by assuming that $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$ and pick an element $D \in \mathfrak{D}_{\mathfrak{C}}$. We wish to show that $\{W_T^{\pi} \,|\, \pi \in \Pi_{\mathfrak{C}}\}$ is bounded in probability. Since $D_T > 0$, this is equivalent to showing that $\{D_T W_T^{\pi} \,|\, \pi \in \Pi_{\mathfrak{C}}\}$ is bounded in probability. This easily follows from the fact that DW^{π} for $\pi \in \Pi_{\mathfrak{C}}$ are positive supermartingales with $D_0 W_0^{\pi} = 1$ and so, for all $m > 0$, $\sup_{\pi \in \Pi_{\mathfrak{C}}} \mathbb{P}[D_T W_T^{\pi} > m] \leq m^{-1} \sup_{\pi \in \Pi_{\mathfrak{C}}} \mathbb{E}[D_T W_T^{\pi}] \leq m^{-1}$.

The implication (3) $\Rightarrow$ (4) is (as already noticed) trivial.

For (4) $\Rightarrow$ (5), if $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ then Lemma 2.5 shows that NUIP$_{\mathfrak{C}}$ fails.

The implication (5) $\Rightarrow$ (1) is significantly harder; after some preparation in the sequel, its proof will be the context of Lemma 4.1 in the next section.

Finally, the claim for the further equivalences in the cone-constrained case is obvious. $\square$

Remark 3.6. Unless $\mathfrak{C}$ is a cone, the conditions of Theorem 3.5 are *not* equivalent to NA$_{\mathfrak{C}}$ in general. Actually, an increasing (but not unbounded) profit might exist. Indeed, in the context of Example 3.1 consider the constraints set $\mathfrak{C} = [0,1] \times [0,1]$. Since $\check{\mathfrak{C}} = \{0\}$, NUIP$_{\mathfrak{C}}$ trivially holds, but of course $\pi = (1,0) \in \mathfrak{C}$ is an increasing profit.

3.5. **No-free-lunch equivalences in the infinite-time horizon case.** The situation for infinite-time horizon exponential Lévy models is drastically different than what we have seen in Theorems 2.7 and 3.5. It turns out that we can *always* construct free lunches (albeit not increasing profit necessarily) unless the *original* measure $\mathbb{P}$ is supermartingale measure, meaning that W^{π} is a $\mathbb{P}$-supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$.

Previous definitions on free lunches, equivalent (super)martingale measures and deflators can be read for infinite-time horizons by plugging $T = +\infty$; the terminal wealths W_T^{π} in Definition 2.1 have to be replaced by $W_{\infty}^{\pi} = \lim_{t \to \infty} W_t^{\pi}$, where we assume that this last limit exists $\mathbb{P}$-a.s. (this is for example the case when π is supported on a stochastic interval $[\![0, \tau]\!]$, where τ is a $\mathbb{P}$-a.s. finite stopping time).

Theorem 3.7. *For an exponential Lévy stock-price model under closed convex constraints* $\mathfrak{C} \subseteq \mathbb{R}^d$ *on a infinite-time horizon, the following are equivalent:*

(1) W^{π} *is a* $\mathbb{P}$*-supermartingale for all* $\pi \in \Pi_{\mathfrak{C}}$.
(2) *An* ESMM$_{\mathfrak{C}}$ *exists:* $\mathfrak{Q}_{\mathfrak{C}} \neq \emptyset$;
(3) *An* ESMD$_{\mathfrak{C}}$ *exists:* $\mathfrak{D}_{\mathfrak{C}} \neq \emptyset$;
(4) *The* NFLVR$_{\mathfrak{C}}$ *condition holds;*
(5) *The* NUPBR$_{\mathfrak{C}}$ *condition holds;*

(6) *The* $\mathrm{NA}_{\mathfrak{C}}$ *condition holds.*

Remark 3.8. Even though there is no *direct* reference to a condition involving the Lévy triplet (b, c, ν) as there was in Theorems 2.7 and 3.5 for finite-time horizons, observe that actually condition (1) of Theorem 3.7 is one. Indeed, in order for $\mathbb{P}$ to be such that W^π is a $\mathbb{P}$-supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$ it is necessary and sufficient that $\mathbb{E}[\mathrm{p}^\top X_1] \leq 0$ (this does not mean that $\mathrm{p}^\top X_1$ is integrable — just that the positive part is integrable) for all $\mathrm{p} \in \mathfrak{C} \cap \mathfrak{C}_0$. In other words, for every $\mathrm{p} \in \mathfrak{C} \cap \mathfrak{C}_0$ we must have $\mathrm{p}^\top b + \int \mathrm{p}^\top x \mathbb{I}_{\{|x|>1\}} \nu(\mathrm{d}x) \leq 0$.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) are all trivial. We only prove (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (1) below by showing that if $\mathbb{P}$ is not a supermartingale measure, both $\mathrm{NUPBR}_{\mathfrak{C}}$ and $\mathrm{NA}_{\mathfrak{C}}$ fail.

Assume then that $\mathbb{P}$ is not a supermartingale measure. If $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$, then $\mathrm{NUIP}_{\mathfrak{C}}$ fails and so both $\mathrm{NUPBR}_{\mathfrak{C}}$ and $\mathrm{NA}_{\mathfrak{C}}$ will fail. On the other hand, if $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, the numéraire portfolio exists: it is a constant portfolio ρ that gives rise to a positive supermartingale $(W^\rho)^{-1}$. We know that $(W^\rho_\infty)^{-1} := \lim_{t\to\infty} (W^\rho_t)^{-1}$ exists $\mathbb{P}$-a.s. in $\mathbb{R}_+$. We actually claim that $(W^\rho_\infty)^{-1} = 0$. Indeed, the fact that this limit is a constant follows from Kolmogorov's 0-1 law for the Lévy process $L^\rho := \log W^\rho$; but we can only have $L^\rho_\infty = +\infty$, for otherwise L^ρ would be a Lévy process with finite limit at infinity, which cannot happen unless it is identically constant zero, and this would mean $W^\rho \equiv 1$, or $\rho \in \mathfrak{N}$ which cannot happen unless $\mathbb{P}$ is a supermartingale measure (see Example 3.3) and we are working under the assumption that it is not. Now, the fact $W^\rho_\infty = \infty$ allows us to construct portfolios $\pi_n \in \Pi_{\mathfrak{C}}$ by requiring $\pi_n := \rho \mathbb{I}_{[\![0,\tau_n]\!]}$, where τ_n is the finite stopping time $\tau_n := \inf\{t \in \mathbb{R}_+ \mid W^\rho_t \geq n\}$. Then, $W^{\pi_n}_\infty \geq n$ and both conditions $\mathrm{NUPBR}_{\mathfrak{C}}$ and $\mathrm{NA}_{\mathfrak{C}}$ fail. □

Remark 3.9. (On the One-Dimensional, Unconstrained Case). For the infinite time-horizon case, Selivanov [26] shows that if $d = 1$ and $\mathfrak{C} = \mathbb{R}^d$, then NFLVR is equivalent to the following: either (1) S is a $\mathbb{P}$-martingale, or (2) S is a $\mathbb{P}$-supermartingale and the jumps of S are locally unbounded above. We can actually get this result from Theorem 3.7: if the jumps of S are locally bounded above (equivalently, the jumps of X are bounded above) we have that 0 belongs to the relative interior of the natural constraints $\mathfrak{C}_0$. From Remark 3.8 this would mean that both $\mathbb{E}[X_1] \leq 0$ and $\mathbb{E}[-X_1] \leq 0$, which means that X, and thus S, is a $\mathbb{P}$-martingale.

3.6. **Relative rate of return.** In order to figure out whether a *constant* vector $\rho \in \mathfrak{C}$ is the numéraire portfolio we should (at least) check that W^π / W^ρ is a supermartingale for all other constant $\pi \in \mathfrak{C}$. This is seemingly weaker than the requirement of Definition 3.2, but the two will actually turn out to be equivalent.

Since for all π and ρ vectors in $\mathfrak{C}$ we have that W^π and W^ρ are exponential Lévy process we get that the log-relative-wealth-process $L^{\pi|\rho} := \log(W^\pi / W^\rho)$ is a Lévy process itself. The exponential formula (0.4) implies that $\mathbb{E}[W^\pi_T / W^\rho_T] = \mathbb{E} \exp(L^{\pi|\rho}_T) = \exp\big(T \mathfrak{rel}(\pi \mid \rho)\big)$,

where straightforward computations lead us to set

$$\mathfrak{rel}(\pi \,|\, \rho) \;:=\; (\pi-\rho)^\top b - (\pi-\rho)^\top c\rho + \int \left[\frac{(\pi-\rho)^\top x}{1+\rho^\top x} - (\pi-\rho)^\top x \mathbb{I}_{\{|x|\leq 1\}} \right] \nu(\mathrm{d}x). \tag{3.1}$$

The quantity $\mathfrak{rel}(\pi \,|\, \rho)$ is the *relative rate of return* of π with respect to ρ.

The integrand appearing in (3.1) is equal to $(1+\pi^\top x)/(1+\rho^\top x) - 1 - (\pi-\rho)^\top x \mathbb{I}_{\{|x|\leq 1\}}$; this quantity is bounded from below by -1 on $\{|x|>1\}$ for the Lévy measure ν, while on $\{|x|\leq 1\}$ behaves like $(\rho-\pi)^\top x x^\top \rho$, which is comparable to $|x|^2$. It follows that the integral always makes sense, but can take the value $+\infty$. In any case, the quantity $\mathfrak{rel}(\pi \,|\, \rho)$ of (3.1) is well-defined.

The relative wealth process W^π/W^ρ is a supermartingale if and only if $\mathbb{E}[W^\pi_T/W^\rho_T] \leq 1$, equivalently if $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$. We remark that this result extends to the case where π (and ρ) are non-constant predictable processes in $\Pi_{\mathfrak{C}}$; the reason being that the predictable finite variation part of $W^\pi/W^\rho = \exp(L^{\pi|\rho})$ — given that it is a special semimartingale and admits a Doob-Meyer decomposition — is $\int_0^\cdot \exp(L^{\pi|\rho}_{t-}) \mathfrak{rel}(\pi_t \,|\, \rho_t) \mathrm{d}t$, one can check this directly or refer to Karatzas and Kardaras [21]. The previous discussion proves the following.

Lemma 3.10. *In order for a constant vector $\rho \in \mathfrak{C}$ to be the numéraire portfolio in the class $\Pi_{\mathfrak{C}}$ it is necessary and sufficient that $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$ for every $\pi \in \mathfrak{C}$.*

It follows then that in order to prove the implication (5) $\Rightarrow$ (1) in Theorem 3.5 it suffices to show that $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ implies that there exists a $\rho \in \mathfrak{C}$ such that $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$ for every $\pi \in \mathfrak{C}$; this is taken on in Lemma 4.1.

3.7. **The growth-optimal portfolio.** In this subsection we continue towards the goal to construct the numéraire portfolio via the Lévy triplet (b, c, ν) in case $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, using the fact that it is *essentially* equal to the growth-optimal portfolio, which has been studied in Algoet and Cover [1] in a general discrete-time setting. Take a constant portfolio $\pi \in \Pi_{\mathfrak{C}}$; its *growth rate* is defined as the drift rate of the log-wealth process $\log W^\pi$. Since $\log W^\pi$ is a Lévy process, one can use (0.3) and formally (since it will not always exist) compute the growth rate of π to be

$$\mathfrak{g}(\pi) := \pi^\top b - \frac{1}{2}\pi^\top c \pi + \int \left[\log(1+\pi^\top x) - \pi^\top x \mathbb{I}_{\{|x|\leq 1\}} \right] \nu(\mathrm{d}x). \tag{3.2}$$

It turns out that the numéraire portfolio and the *growth-optimal portfolio* (defined as the one that maximizes the growth rate (3.2) over all portfolios) are essentially the same.

Example 3.11. We consider the Samuelson-Black-Scholes-Merton model $X_t = bt + \sigma\beta_t$, in the unconstrained case $\mathfrak{C} = \mathbb{R}^d$. According to Example 2.4 we have $\mathfrak{I} = \emptyset$ if and only if there exists $\rho \in \mathbb{R}^d$ such that $b = c\rho$ (which always holds if $c = \sigma\sigma^\top$ is nonsingular). The derivative of the growth rate is $(\nabla \mathfrak{g})_\pi = b - c\pi$, and it is trivially zero for $\pi \equiv \rho$, which is the numéraire portfolio.

Let us describe in more generality the connection between the numéraire and the growth-optimal portfolio, being somewhat informal for the moment: a vector $\rho \in \mathfrak{C}$ maximizes this concave function $\mathfrak{g}$ if and only if the directional derivative of $\mathfrak{g}$ at the point ρ in the direction of $\pi - \rho$ is negative for any $\pi \in \Pi$. One can use (3.2) to compute $(\nabla \mathfrak{g})_\rho(\pi - \rho)$ and it is straightforward to see that it turns out to be exactly $\mathfrak{rel}(\pi \mid \rho)$ of (3.1).

Let us try now to be a little more formal. We do not know if we can differentiate under the integral appearing in equation (3.2). Even more to the point, we do not know a priori whether the integral is well-defined: both its positive and negative parts could be infinite. Non-integrability of the negative part is not too severe, since one wants to *maximize* $\mathfrak{g}$: if a portfolio π results in an integrand whose negative part integrates to infinity, all vectors $a\pi$ for $a \in [0, 1)$ will lead to a finite result. More problematic is the fact that the *positive* part can integrate to infinity, especially when one notices that if this happens for at least one vector $\pi \in \mathfrak{C}$, concavity will imply that it happens for *many* vectors — actually for *all* vectors in the relative interior of $\mathfrak{C}$, with the possible exception of those of the form $-a\pi$ for $a > 0$. This problem is related to the one when the expected log-utility is infinite and one cannot find a unique solution to the log-utility maximization problem — see the next subsection 3.8.

In the spirit of the above discussion, let us describe a class of Lévy measures for which the concave growth rate function $\mathfrak{g}(\cdot)$ of (3.2) *is* well-defined.

Definition 3.12. A Lévy measure ν *integrates the log*, if $\int \log(1+|x|)\mathbb{I}_{\{|x|>1\}}\nu(\mathrm{d}x) < \infty$. For any Lévy measure ν, a sequence $(\nu_n)_{n\in\mathbb{N}}$ of Lévy measures that integrate the log with $\nu_n \sim \nu$, whose densities $f_n := \mathrm{d}\nu_n/\mathrm{d}\nu$ satisfy $0 < f_n \le 1$, $f_n(x) = 1$ for $|x| \le 1$, and $\lim_{n\to\infty} \uparrow f_n = \mathbb{I}$, will be called an *approximating sequence*.

One specific choice for the densities appearing in the definition of approximating sequence is $f_n(x) = \mathbb{I}_{\{|x|\le 1\}} + |x|^{-1/n}\mathbb{I}_{\{|x|>1\}}$. The sets $\mathfrak{C}_0$, $\mathfrak{N}$ and $\mathfrak{I}$ remain unchanged if we move from the original triplet to any of the approximating triplets, thus $\mathfrak{I}(b, c, \nu) \cap \check{\mathfrak{C}} = \emptyset$ if and only if $\mathfrak{I}(b, c, \nu_n) \cap \check{\mathfrak{C}} = \emptyset$ for all $n \in \mathbb{N}$.

The problem of the positive infinite value for the integral appearing in equation (3.2) disappears when the Lévy measure ν integrates the log, and the growth-optimal portfolio is also the numéraire portfolio. In the general case, where ν might not integrate the log, our strategy will be the following: solve the optimization problem concerning $\mathfrak{g}$ for a sequence of problems using the approximation described in Definition 3.12, and then show that the corresponding solutions converge to the solution of the original problem.

Remark 3.13. Even in the unconstrained case *the supermartingale deflator corresponding to the numéraire portfolio need not be a martingale, and can in fact be a strict supermartingale.* Of course, the importance of supermartingales in utility maximization (after all, we are basically dealing with log utility here) has been recognized by Kramkov and Schachermayer [23]. Hurd [19] gives a treatment of log-utility in exponential Lévy models. For completeness, we give in the next paragraph an elementary example to illustrate what can go wrong.

Take a one-dimensional Lévy process with X with $b = 1$, $c = 0$ and $\nu(\mathrm{d}x) = (1 + x)\mathbb{I}_{(-1,1]}(x)\mathrm{d}x$. One can easily check that $\mathfrak{C}_0 = [-1, 1]$ and that $\mathfrak{g}'$ (the derivative of $\mathfrak{g}$) is decreasing in $\pi \in (-1, 1)$ with $\mathfrak{g}'(-1) = +\infty$ and $\mathfrak{g}'(1) = 1/3$. The numéraire portfolio is $\rho = 1$ and $(W^\rho)^{-1}$ is a strict Lévy supermartingale, since $\mathfrak{rel}(0 \,|\, 1) = -\mathfrak{g}'(1) = -1/3 < 0$.

The above fact gives some justice to the Esscher transform method in the proof of Theorem 2.7, which provides us with a probability measure. The situation should be contrasted to the continuous-path case of Example 3.11 where, in the absence of constraints, $(W^\rho)^{-1}$ is a martingale. We also see that we cannot expect to be able in general to compute the numéraire portfolio just by naively trying to solve $\nabla \mathfrak{g}(\rho) = \mathfrak{rel}(0 \,|\, \rho) = 0$.

3.8. **Relative log-optimality and the numéraire portfolio.** We rush through the (well-understood) relevance of the numéraire portfolio with the *relatively log-optimal*, i.e., a portfolio $\rho \in \Pi_\mathfrak{C}$ such that $\mathbb{E}[\log(W^\pi_T / W^\rho_T)] \leq 0$ (here, it is tacitly assumed that $\mathbb{E}\log^+(W^\pi_T / W^\rho_T) < \infty$), for every $\pi \in \Pi_\mathfrak{C}$. A treatment for the general semimartingale case is given in Karatzas and Kardaras [21].

If the numéraire portfolio ρ exists, then for any other $\pi \in \Pi_\mathfrak{C}$ we have $\mathbb{E}[W^\pi_T / W^\rho_T] \leq 1$; applying Jensen's inequality we get $\mathbb{E}\log(W^\pi_T / W^\rho_T) \leq 0$, i.e., that ρ is relatively log-optimal.

Now, suppose that the numéraire portfolio does not exist — according to Theorem 3.5, this means that we can pick $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$. For any $\rho \in \Pi_\mathfrak{C}$, we have $\rho + \xi \in \Pi_\mathfrak{C}$ as well; simple computations, using the fact that $\xi \in \mathfrak{I}$, give that the relative-log-ratio $\log(W^{\rho+\xi}_T / W^\rho_T)$ is equal to $(\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x|\leq 1\}} \nu(\mathrm{d}x))T + \sum_{0 \leq t \leq T} \log[1 + \xi^\top \Delta X_t / (1 + \rho_t^\top \Delta X_t)]$, which by Definition 2.3 of immediate arbitrage opportunities is positive, with positive probability of being strictly positive; this implies $\mathbb{E}\log(W^{\rho+\xi}_T / W^\rho_T) > 0$. Thus, if the numéraire portfolio does not exist, a relative-log-optimal portfolio cannot exist either.

The somewhat amazing conclusion from the above discussion above is that for $\rho \in \Pi_\mathfrak{C}$ we have the following equivalence:

$$\log\left(\mathbb{E}\frac{W^\pi_T}{W^\rho_T}\right) \leq 0, \quad \text{for all } \pi \in \Pi_\mathfrak{C} \quad \Longleftrightarrow \quad \mathbb{E}\log\left(\frac{W^\pi_T}{W^\rho_T}\right) \leq 0, \quad \text{for all } \pi \in \Pi_\mathfrak{C}.$$

Of course, Jensen's inequality gives direction $\Rightarrow$ for any portfolios π and ρ in $\Pi_\mathfrak{C}$; the opposite direction $\Leftarrow$ fails in general for any π and ρ in $\Pi_\mathfrak{C}$ — it *will* hold for all $\pi \in \Pi_\mathfrak{C}$ if we fix the *specific* ρ that makes *all* expectations of the relative log-wealth process nonpositive.

If for the relative log-optimal portfolio ρ we have $\mathbb{E}\log W^\rho_T < \infty$, then ρ also is the unique log-optimal portfolio. If $\mathbb{E}\log W^\rho_T = \infty$, the log-utility maximization problem has an infinite number of solutions. For an example where this happens take a one-dimensional Lévy process with $b = c = 0$ and a Lévy measure with density $\nu(\mathrm{d}x)/\mathrm{d}x = \mathbb{I}_{(-1,1]}(x) + x^{-1}(\log(1 + x))^{-2}\mathbb{I}_{[1,\infty)}(x)$ — we have $\mathfrak{C}_0 = [0, 1]$ and it is easy to check that $\mathbb{E}[\log W^\pi_T] = \infty$ for all $\pi \in (0, 1)$. For this example, the problem of maximizing expected log-utility does not have unique solution. Of course, the numéraire and relatively log-optimal portfolios exist and will be unique (and the same).

4. Finishing the Proof of Theorem 3.5

The focus of this section is the proof of the following Lemma 4.1 which will complete the proof of Theorem 3.5. We state it separately of everything else because it will also find good use in Karatzas and Kardaras [21].

Lemma 4.1. *Let (b, c, ν) be a Lévy triplet and $\mathfrak{C}$ a closed convex subset of $\mathbb{R}^d$. Then,* $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ if and only if *there exists a unique vector $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ with $\nu[\rho^\top x \leq -1] = 0$ such that $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.*

If ν integrates the log, the vector ρ above is characterized as $\rho = \arg\max_{\pi \in \mathfrak{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$. In general, ρ is the limit of solutions to a series of problems, in which ν is replaced by a sequence of approximating measures.

Although it will come as a result of Theorem 3.5, let us give a quick proof of the fact that if $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$ then one cannot find a $\rho \in \mathfrak{C}$ such that $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$ for all $\pi \in \mathfrak{C}$. To this end, pick a vector $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$, and suppose that ρ satisfied $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$, for all $\pi \in \mathfrak{C}$. Since $\xi \in \check{\mathfrak{C}}$, we have $n\xi \in \mathfrak{C}$ for all $n \in \mathbb{N}$ and the convex combination $(1 - n^{-1})\rho + \xi \in \mathfrak{C}$ too; but $\mathfrak{C}$ is closed, and so $\rho + \xi \in \mathfrak{C}$. Easy computations show that $\mathfrak{rel}(\rho + \xi \,|\, \rho)$ is equal to $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) + \int [\xi^\top x / (1 + \rho^\top x)] \nu(\mathrm{d}x)$; this is strictly positive quantity from the definition of ξ. This is a contradiction to ρ satisfying $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.

We want to prove the converse; namely if $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$, then one can find a ρ that satisfies the requirement of Lemma 4.1 — subsections 4.1 and 4.2 are devoted to the proof of this. In the process we shall need the following simple characterization of the condition $\mathfrak{I} \cap \check{\mathfrak{C}} \neq \emptyset$:

Lemma 4.2. *If $\mathfrak{C} \subseteq \mathfrak{C}_0$ and $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$, then $\xi \in \mathfrak{I}$ if and only if $\mathfrak{rel}(0 \,|\, a\xi) \leq 0$ for all $a \in \mathbb{R}_+$.*

Proof. The fact that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$ implies $\mathfrak{rel}(0 \,|\, a\xi) \leq 0$ for all $a \in \mathbb{R}_+$ is trivial.

For the converse, let $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$ satisfy $\mathfrak{rel}(0 \,|\, a\xi) \leq 0$ for all $a \in \mathbb{R}_+$; we wish to show that $\xi \in \mathfrak{I}$. The second condition of Definition 2.3 is readily satisfied, since we assume that $\mathfrak{C}$ contains the natural constraints. Now, for all $a \in \mathbb{R}_+$, we have $-a^{-1}\mathfrak{rel}(0 \,|\, a\xi) \geq 0$; writing this down gives $\xi^\top b - a\xi^\top c \xi + \int [\xi^\top x / (1 + a\xi^\top x) - \xi^\top x \mathbb{I}_{\{|x| \leq 1\}}] \nu(\mathrm{d}x) \geq 0$. Observe that the integrand $\xi^\top x / (1 + a\xi^\top x) - \xi^\top x \mathbb{I}_{\{|x| \leq 1\}}$ is ν-integrable and decreasing in a (remember that $\nu[\xi^\top x < 0] = 0$), so we must have $\xi^\top c = 0$ (condition (1) of Definition 2.3), which now implies that $\xi^\top b + \int [\xi^\top x / (1 + a\xi^\top x) - \xi^\top x \mathbb{I}_{\{|x| \leq 1\}}] \nu(\mathrm{d}x) \geq 0$. Letting $a \to \infty$ and using the dominated convergence theorem and we get condition (3) of Definition 2.3, namely $\xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(\mathrm{d}x) \geq 0$. □

We make one more observation. On several occasions during the course of the proof we shall use Fatou's lemma in the following form: if we are given a *finite* measure κ and a sequence $(v_n)_{n \in \mathbb{N}}$ of Borel-measurable functions that are κ-uniformly bounded from below, then $\int \liminf_{n \to \infty} v_n(x) \kappa(\mathrm{d}x) \leq \liminf_{n \to \infty} \int v_n(x) \kappa(\mathrm{d}x)$. The finite measures κ

that we shall consider will be of the form $(|x| \wedge k)^2 \, \nu(\mathrm{d}x)$, where $k \in \mathbb{R}_+$ and ν is our Lévy measure.

We can now proceed with the proof of the sufficiency of the condition $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$ in solving $\mathfrak{rel}(\pi \,|\, \rho) \leq 0$. We shall first do so for the case of a Lévy measure that integrates the log, then extend to the general case. Throughout the course of the proof we shall be assuming that $\mathfrak{C} \subseteq \mathfrak{C}_0$; otherwise, replace $\mathfrak{C}$ by $\mathfrak{C} \cap \mathfrak{C}_0$.

4.1. **Proof of Lemma 4.1 for a Lévy measure that integrates the log.** We are trying to show (1) $\Rightarrow$ (2) of Lemma 4.1, so let us assume $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$. For this subsection we also make the assumption $\int_{\{|x|>1\}} \log(1+|x|)\nu(\mathrm{d}x) < \infty$.

Recall from subsection 3.7 the growth rate function $\mathfrak{g}$ of (3.2). This is a concave function on $\mathfrak{C}$, it is well-defined, in the sense that we always have $\mathfrak{g}(\pi) < +\infty$ for $\pi \in \mathfrak{C}$ and upper semi-continuous on $\mathfrak{C}$ (the last two facts follow because ν integrates the log). Of course, $\mathfrak{g}$ can take the value $-\infty$ on the boundary of $\mathfrak{C}$.

Set $\mathfrak{g}_* := \sup_{\pi \in \mathfrak{C}} \mathfrak{g}(\pi)$, and let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of vectors in $\mathfrak{C}$ with $\lim_{n \to \infty} \mathfrak{g}(\rho_n) = \mathfrak{g}_*$. Since for any $\pi \in \mathfrak{C}$ and any $\zeta \in \mathfrak{N}$ we have $\mathfrak{g}(\pi + \zeta) = \mathfrak{g}(\pi)$, we can choose the sequence ρ_n to take values on the subspace $\mathfrak{N}^\perp$ (it would be useful to recall the discussion of Remark 1.4).

We first want to show that the sequence $(\rho_n)_{n \in \mathbb{N}}$ of vectors of $\mathfrak{C} \cap \mathfrak{N}^\perp$ is bounded; then we shall be able to pick a convergent subsequence. Suppose then on the contrary that $(\rho_n)_{n \in \mathbb{N}}$ unbounded, and without loss of generality suppose also that the sequence of unit-length vectors $\xi_n := \rho_n / |\rho_n|$ converges to a unit-lenth vector $\xi \in \mathfrak{N}^\perp$ (picking a subsequence otherwise). We shall use Lemma 4.2 applied to the vector ξ and show that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$, contradicting condition (1) of Lemma 4.1.

Start by picking any $a \in \mathbb{R}_+$; for all large enough $n \in \mathbb{N}$ we have $a\xi_n \in \mathfrak{C}$, and since $\mathfrak{C}$ is closed we have $a\xi \in \mathfrak{C}$ as well, which implies $\xi \in \check{\mathfrak{C}}$ (since $a \in \mathbb{R}_+$ is arbitrary). We have $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$, and only need to show $\mathfrak{rel}(0 \,|\, a\xi) \leq 0$. For this, we can assume that the sequence $(\rho_n)_{n \in \mathbb{N}}$ is picked in such a way that the functions $[0,1] \ni u \mapsto \mathfrak{g}(u\rho_n)$ are increasing; otherwise, replace ρ_n by the vector $u\rho_n$ for the choice of $u \in [0,1]$ that maximizes $[0,1] \ni u \mapsto \mathfrak{g}(u\rho_n)$. This would imply that eventually, for all large enough $n \in \mathbb{N}$ we have $\mathfrak{rel}(0 \,|\, a\xi_n) \leq 0$; this means

$$\int \left[\frac{-\xi_n^\top x}{1 + a\xi_n^\top x} + \xi_n^\top x \mathbb{I}_{\{|x| \leq 1\}} \right] \nu(\mathrm{d}x) \leq \xi_n^\top b - a\xi_n^\top c \xi_n.$$

If we can show that we can apply Fatou's lemma to the quantity on the left-hand-side of this inequality, we get the same inequality with ξ in place of ξ_n and so $\mathfrak{rel}(0 \,|\, a\xi) \leq 0$; an application of Lemma 4.2 shows that $\xi \in \mathfrak{I} \cap \check{\mathfrak{C}}$, contradicting condition (1) of Lemma 4.1.

To show that we can apply Fatou's lemma, let us show that the integrand is bounded from below for the finite measure $(|x| \wedge k)^2 \, \nu(\mathrm{d}x)$ with $k := 1 \wedge (2a)^{-1}$. Since $\xi_n^\top x / (1 + a\xi_n^\top x) \leq a^{-1}$ and $\left|\xi_n^\top x\right| \leq |x|$, the integrand is uniformly bounded from below by $-(a^{-1} + 1)$, and we only need consider what happens on the set $\{|x| \leq k\}$; there, the integrand is equal to $-a(\xi_n^\top x)^2 / (1 + a\xi_n^\top x)$, which cannot be less than $-2a|x|^2$ and we are done.

We now know that $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^d$; without loss of generality, suppose that $(\rho_n)_{n\in\mathbb{N}}$ converges to a point $\rho \in \mathfrak{C}$ (otherwise, choose a convergent subsequence). The concavity of $\mathfrak{g}$ implies that $\mathfrak{g}_*$ is a finite number and it is obvious from continuity that $\mathfrak{g}(\rho) = \mathfrak{g}_*$. Of course, we have that $\nu\left[\rho^\top x \le -1\right] = 0$, otherwise $\mathfrak{g}(\rho) = -\infty$.

Pick now any $\pi \in \mathfrak{C}^\diamond := \{\pi \in \mathfrak{C} \,|\, \nu[\pi^\top x \le -u] = 0 \text{ for some } u < 1\}$, then it is clear that $g(\pi) > -\infty$. If follows that the mapping $[0,1] \ni u \mapsto \mathfrak{g}(\rho + u(\pi - \rho))$ is well-defined (i.e., real-valued), concave and decreasing, so that the right-derivative at $u = 0$ should be negative; this derivative is just $\mathfrak{rel}(\pi \,|\, \rho)$, so we have $\mathfrak{rel}(\pi \,|\, \rho) \le 0$ for $\pi \in \mathfrak{C}^\diamond$.

The extension of the inequality $\mathfrak{rel}(\pi \,|\, \rho) \le 0$ for all $\pi \in \mathfrak{C}$ now follows easily. Indeed, if $\pi \in \mathfrak{C}$, then for $0 \le u < 1$ we have $u\pi \in \mathfrak{C}^\diamond$ and $\mathfrak{rel}(u\pi \,|\, \rho) \le 0$; by using Fatou's lemma one can easily check that we also have $\mathfrak{rel}(\pi \,|\, \rho) \le 0$. □

4.2. **The extension to general Lévy measures.** We now have to extend the result of the previous subsection to the case where ν does not necessarily integrate the log. Recall from Definition 3.12 the use of the approximating triplets (b, c, ν_n), where for every $n \in \mathbb{N}$ we define the measure $\nu_n(\mathrm{d}x) := f_n(x)\nu(\mathrm{d}x)$; all these measures integrate the log. We assume throughout that $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

We remarked that the sets $\mathfrak{N}$ and $\mathfrak{I}$ remain invariant if we change the Lévy measure from ν to ν_n. Then, since we have $\mathfrak{I}(b, c, \nu_n) \cap \check{\mathfrak{C}} = \emptyset$, the discussion in the previous section, gives us unique vectors $\rho_n \in \mathfrak{C} \cap \mathfrak{N}^\perp$ such that $\mathfrak{rel}_n(\pi \,|\, \rho_n) \le 0$ for all $\pi \in \mathfrak{C}$, where $\mathfrak{rel}_n$ is associated with the triplet (b, c, ν_n).

As before, the constructed sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded. To prove it, we shall use Lemma 4.2 again, in the exact same way that we did for the case of a measure that integrates the log. Assume by way of contradiction that $(\rho_n)_{n\in\mathbb{N}}$ is not bounded. By picking a subsequence if necessary, assume without loss of generality that $|\rho_n|$ diverges to infinity. Now, call $\xi_n := \rho_n/|\rho_n|$. Again, by picking a further subsequence if the need arises, assume that $\lim_{n\to\infty} \xi_n = \xi$, where ξ is a unit vector in $\mathfrak{N}^\perp$. Since $\rho_n \in \mathfrak{C}$ for all $n \in \mathbb{N}$ it follows that $a\xi \in \mathfrak{C}$ for all $a \in \mathbb{R}_+$, i.e., $\xi \in \check{\mathfrak{C}} \setminus \mathfrak{N}$. We know that for sufficiently large $n \in \mathbb{N}$, we have that $\mathfrak{rel}_n(0 \,|\, a\xi_n) \le 0$; equivalently $\int [-\xi_n^\top x f_n(x)/(1 + a\xi_n^\top x) + \xi_n^\top x \mathbb{I}_{\{|x|\le 1\}}]\nu(\mathrm{d}x) \le \xi_n^\top b - a\xi_n^\top c\xi_n$. The situation is exactly the same as in the proof in the case of a measure that integrates the log, but for the appearance of the density $f_n(x)$ which can only have a positive effect on any lower bounds that we have established there, since $0 < f_n \le 1$. We show that the integrand is bounded from below for the finite measure $(|x| \wedge k)^2\, \nu(\mathrm{d}x)$ with $k = 1 \wedge (2a)^{-1}$, thus we can apply Fatou's lemma to the left-hand-side of this inequality to get the same inequality with ξ in place of ξ_n, and so $\mathfrak{rel}(0 \,|\, a\xi) \le 0$. Invoking Lemma 4.2, we arrive at a contradiction with the assumption $\mathfrak{I} \cap \check{\mathfrak{C}} = \emptyset$.

Now that we know that $(\rho_n)_{n\in\mathbb{N}}$ is a bounded sequence, we can assume that it converges to a point $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$, picking a subsequence if needed. We shall show that ρ satisfies $\mathfrak{rel}(\pi \,|\, \rho) \le 0$ for all $\pi \in \mathfrak{C}$. Pick any $\pi \in \mathfrak{C}$; we know that we have

$$\int \left[\frac{(\pi - \rho_n)^\top x}{1 + \rho_n^\top x} f_n(x) - (\pi - \rho_n)^\top x \mathbb{I}_{\{|x|\le 1\}} \right] \nu(\mathrm{d}x) \le -(\pi - \rho_n)^\top b + (\pi - \rho_n)^\top c \rho_n$$

for all $n \in \mathbb{N}$. Yet once more, we shall use Fatou's lemma on the left-hand-side to get to the limit the same inequality with ρ_n and $f_n(x)$ being replaced by ρ and 1 respectively; in other words, we get $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$.

To justify the use of Fatou's lemma, we shall show that the integrands are uniformly bounded from below for the finite measure $(|x| \wedge k)^2 \nu(\mathrm{d}x)$, where $k := 1 \wedge (2 \sup_{n \in \mathbb{N}} |\rho_n|)^{-1}$ is a strictly positive number from the boundedness of $(\rho_n)_{n \in \mathbb{N}}$. First, observe that the integrands are uniformly bounded by $-1 - \sup_{n \in \mathbb{N}} |\pi - \rho_n|$, which is a finite number. Thus, we only need worry about the set $\{|x| \leq k\}$. There, the integrands are equal to $(\pi - \rho_n)^\top x (\rho_n^\top x)/(1 + \rho_n^\top x)$; this cannot be less than $-2 \sup_{n \in \mathbb{N}} (|\pi - \rho_n| |\rho_n|) |x|^2$, and Fatou's lemma can be used.

Up to now we have shown that $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ for the limit ρ of a subsequence of $(\rho_n)_{n \in \mathbb{N}}$. Nevertheless, carrying the previous steps we see that *every* subsequence of $(\rho_n)_{n \in \mathbb{N}}$ has a further convergent subsequence whose limit $\hat{\rho} \in \mathfrak{C} \cap \mathfrak{N}^\perp$ satisfies $\mathfrak{rel}(\pi \mid \hat{\rho}) \leq 0$ for all $\pi \in \mathfrak{C}$. The uniqueness of $\rho \in \mathfrak{C} \cap \mathfrak{N}^\perp$ that satisfies $\mathfrak{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \mathfrak{C}$ gives that $\hat{\rho} = \rho$, and we conclude that the whole sequence $(\rho_n)_{n \in \mathbb{N}}$ converges to ρ. □

Appendix A. Facts Regarding Lévy Processes

We hereby collect some results that are used within the text; they are mostly simple consequences of the definition of a Lévy process; we include them for completeness, since they might not be part of the usual treatment in textbooks.

First of all, Lévy process have the following property, which already points out in some way the fact that "if there is arbitrage it should be an *increasing* profit":

Lemma A.1. *If for some one-dimensional Lévy process L and some time $T > 0$ we have $L_T \geq 0$, $\mathbb{P}$-a.s., then L is actually an increasing process.*

Proof. Write $L_T = L_{T/2} + L'_{T/2}$, where $L'_{T/2}$ is independent of, and has the same distribution as $L_{T/2}$. Then, $0 = \mathbb{P}[L_T < 0] = \mathbb{P}[L_{T/2} < -L'_{T/2}] \geq \mathbb{P}[L_{T/2} < 0, L'_{T/2} < 0] = (\mathbb{P}[L_{T/2} < 0])^2$, hence $\mathbb{P}[L_{T/2} < 0] = 0$. Continuing like this and using the stationary-increments property of L we get $\mathbb{P}[L_t < 0] = 0$ for all $t \in \mathbb{D} := \{kT/2^n \mid n \in \mathbb{N},\ k = 0, \ldots, 2^n\}$. The stationarity of increments of L couple with the countability of $\mathbb{D}$ implies that $\mathbb{P}[L_s \leq L_t$ for all $s \in \mathbb{D},\ t \in \mathbb{D}$ with $s < t] = 1$; then, right-continuity of L will give us that the latter is an increasing process. □

A $\mathbf{F}$-Lévy process X is *regenerating* at every stopping time σ — this means that on $\{\sigma < \infty\}$ the process $Y := (X_{\sigma+s} - X_\sigma)_{s \in \mathbb{R}_+}$ is an $\mathbf{G}$-Lévy process, independent of $\mathcal{F}_\sigma$, where we set $\mathcal{G}_s := \mathcal{F}_{\sigma+s}$ for all $s \in \mathbb{R}_+$. If τ is an $\mathbf{F}$-stopping time with $\sigma \leq \tau$, $\mathbb{P}$-a.s., then the random time $\tau - \sigma$ is an $\mathbf{G}$-stopping time and we obviously have $\Delta Y_{\tau-\sigma} = \Delta X_\tau \mathbb{I}_{\{\sigma < \tau\}}$. These remarks will be used in the proof of the result below which states that the jump-size at a stopping time is independent of whatever has happened *strictly* before that stopping time. This "strict history" notion is formalized by introducing the σ-algebra

$\mathcal{F}_{\tau-}$ of events strictly prior to τ, that is the smallest σ-algebra generated by the class $\mathcal{A}_{\tau-} := \mathcal{F}_0 \cup \{B \cap \{t < \tau\} \,|\, B \in \mathcal{F}_t \text{ for some } t \in \mathbb{R}_+\}$.

Lemma A.2. *If X is an $\mathbf{F}$-Lévy process for some filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, then for any stopping time τ, the jump $\Delta X_\tau \mathbb{I}_{\{\tau < \infty\}}$ is independent of $\mathcal{F}_{\tau-}$.*

Proof. The class $\mathcal{A}_{\tau-}$ defined above is closed under intersection and generates $\mathcal{F}_{\tau-}$. Therefore, it suffices to prove that all $A \in \mathcal{A}_{\tau-}$ are independent of ΔX_τ. For $A \in \mathcal{F}_0$ this is trivial. Thus, consider $A = B \cap \{t < \tau\}$ for some $B \in \mathcal{F}_t$. Let $\sigma := \tau \wedge t$; we have $\sigma \le \tau$ and the regenerating property of Lévy processes implies that $Y := (X_{\sigma+s} - X_\sigma)_{s \in \mathbb{R}_+}$ is an $\mathbf{G}$-Lévy process, independent of $\mathcal{F}_\sigma$, where again $\mathcal{G}$ was defined above. These considerations give us that

$$\mathbb{P}[A \cap \{\Delta X_\tau \in D\}] = \mathbb{P}[B \cap \{t < \tau\} \cap \{\Delta Y_{\tau-\sigma} \in D\}] = \mathbb{P}[B \cap \{t < \tau\}]\mathbb{P}[Y_{\tau-\sigma} \in D]$$

for all $D \in \mathcal{B}(\mathbb{R}^d)$; the last term above is just $\mathbb{P}[A]\mathbb{P}[\Delta X_\tau \in D]$, and the claim follows. □

If the Lévy measure ν of the Lévy process X has finite mass ($\nu(\mathbb{R}^d) < \infty$), then one can represent X in the following form: $X_t = \widetilde{b}t + \sigma\beta_t + \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process with rate $\nu(\mathbb{R}^d)$ and Y_i is a sequence of independent and identically distributed random variables with distribution $\nu(\cdot)/\nu(\mathbb{R}^d)$, further independent of N. In that case we can define the time of the n^{th} jump of X via $\tau_n := \inf\{t \in \mathbb{R}_+ \,|\, N_t = n\}$. The independence of N and $(Y_n)_{n \in \mathbb{N}}$ gives that ΔX_{τ_n} has the distribution of Y_1 and is independent of τ_n. For general stopping times τ with $\mathbb{P}[\Delta X_\tau \neq 0] = 1$ we cannot of course expect that ΔX_τ has the same distribution as Y_1, since we might be sampling the paths in a biased way; for example if D is a Borel subset of $\mathbb{R}^d \setminus \{0\}$ and $\tau := \inf\{t \in \mathbb{R}_+ \,|\, \Delta X_t \in D\}$ then ΔX_τ is only supported on D. Nevertheless, if the decision on whether to stop at the n^{th} jump of X or not is depending *only* on information collected *strictly before* τ_n, the fact that ΔX_τ has the same distribution as Y_1 is still valid.

Lemma A.3. *If the Lévy measure ν of the Lévy process X is such that $\nu(\mathbb{R}^d) < \infty$, and with the notation set above, consider the stopping time $\tau := \bigwedge_{n=1}^\infty (\tau_n)_{A_n}$, where we have set as usual $(\sigma)_A := \sigma\mathbb{I}_A + \infty\mathbb{I}_{\Omega \setminus A}$ for a random time σ and $A \subseteq \Omega$. If $A_n \in \mathcal{F}_{\tau_n -}$ for all $n \in \mathbb{N}$, then, conditional on $\{\tau < \infty\}$, ΔX_τ is identically distributed as Y_1.*

Proof. Observe first of all that we can assume that the sequence $(A_n)_{n \in \mathbb{N}}$ consists of disjoint sets; otherwise, we can replace A_n by $A_n \setminus (\bigcup_{i<n} A_i)$; these sets are still in $\mathcal{F}_{\tau_n -}$, they are disjoint and τ is still given by the same formula $\tau = \bigwedge_{n=1}^\infty (\tau_n)_{A_n}$. We obviously have $\{\tau < \infty\} = \bigcup_{n \in \mathbb{N}} A_n$. Pick any Borel-measurable $g : \mathbb{R}^d \mapsto \mathbb{R}_+$; writing $g(\Delta X_\tau) = \sum_{n=1}^\infty g(\Delta X_{\tau_n})\mathbb{I}_{A_n}$ and observing that the previous Lemma A.2 implies $\mathbb{E}[g(\Delta X_{\tau_n})\mathbb{I}_{A_n}] = \mathbb{E}[g(\Delta X_{\tau_n})]\mathbb{P}[A_n]$ for all $n \in \mathbb{N}$, we get

$$\mathbb{E}[g(\Delta X_\tau)\mathbb{I}_{\{\tau<\infty\}}] = \sum_{n=1}^\infty \mathbb{E}[g(\Delta X_{\tau_n})]\mathbb{P}[A_n] = \sum_{n=1}^\infty \mathbb{E}[g(Y_1)]\mathbb{P}[A_n] = \mathbb{E}[g(Y_1)]\mathbb{P}[\tau < \infty];$$

in other words, $\mathbb{E}[g(\Delta X_\tau) \,|\, \tau < \infty] = \mathbb{E}[g(Y_1)]$, i.e., ΔX_τ is identically distributed as Y_1. □

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